Density correlations in time–dependent Wishart ensemble

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Abstract

Present paper is part of Physics 450 class assignments. I start by reviewing the basic principles of double–line perturbation expansion in random matrix theory. I calculate density of eigenstates of a random Gaussian unitary matrix and derive WIgner's semicircle. Next I calculate density-density correlation function for the density of eigenvalues of random matrix. I consider random matricies drawn from Gaussian Unitary ensemble (GUE), time–dependent GUE, and a so-called complex Wishart ensemble (CWE). I also explain what density of states looks like for CWE and why there is no Wigner's semicircle. I discuss the apparent universalities in the behavior of a correlation function and compare it to the similar quantity for Gaussian unitary ensemble.

Contents

1 Introduction

The outline of the paper is as follows. In section 2 main results from the paper [1] are re-derived from the basic principles and discussed. In particular it is shown that connected density-density correlation of eigenstates of a random matrix ϕ sampled from the so–called time–dependent Gaussian Unitary Ensemble[1] (tGUE).

$$P(\varphi) \propto \exp -\frac{N}{2}\int_{-\infty}^{\infty} dt\, \mathrm{tr}\left[\left(\frac{d\varphi}{dt}\right)^2 + m^2\varphi^2\right] \tag{1.1}$$

is given by the following expression [1, (2.18)]

$$\begin{aligned} \rho_c(\mu,\nu,t) &= \left\langle \frac{1}{N}\,\mathrm{tr}\,\delta(\mu-\varphi(t))\frac{1}{N}\,\mathrm{tr}\,\delta(\nu-\varphi(0))\right\rangle - \\ &\quad - \left\langle \frac{1}{N}\,\mathrm{tr}\,\delta(\mu-\varphi(t))\right\rangle\left\langle \frac{1}{N}\,\mathrm{tr}\,\delta(\nu-\varphi(0))\right\rangle = \\ &= \frac{-m}{8\pi^2N^2\cos\theta\cos\phi}\left\{\frac{1+\cosh m|t|\cos(\theta+\phi)}{[\cosh m|t|+\cos(\theta+\phi)]^2} + \frac{1-\cosh m|t|\cos(\theta-\phi)}{[\cosh m|t|-\cos(\theta-\phi)]^2}\right\}. \end{aligned} \tag{1.2}$$

Here angles θ and ϕ ranging between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ parameterize energy within Wigner's semicircle according to $z = m\sin\theta$ such that eigensates density distribution $\rho(\mu)d\mu = \frac{2}{\pi}\cos^2\theta d\theta$.

In section 3 similar density–density correlator is calculated for the case of time–dependent Complex Wishart Ensemble (tCWE). Thus, generalizing the result of [2].

Mathematically tCWE distribution is given by the same expression (1.1), however now it is the probability density of rectangular matrix α such that $\varphi = \alpha^+\alpha$, while density of states (DOS) is given by an average $\rho(\mu) = \langle\delta(\mu - \alpha^+\alpha)\rangle$ over $P(\alpha)$.

[1] Please note that the factor of N missing in [1, (2.1)] is a misprint.

2 Gaussian Unitary Ensemble

2.1 Wigner's semicircle

Let me start by repeating the well-known results. In the limit of the large matrix size $N \to \infty$ calculating averages over matrix distributions could be easily done in the framework of self–consistent perturbative approach. Probably, the simplest example of such calculation would be the average density of eigenstates, which is known to be given by Wigner semicircle. Let me reproduce it here.

It is useful to introduce Green's function

$$G(z) = \left\langle \frac{1}{z-\varphi} \right\rangle_{\varphi \in \text{GUE}}.$$

According to Sokhotsky formula and translation invariance (in matrix index), the density of states (per node) is then given by

$$\rho(\mu) \equiv \langle \delta(\mu - \varphi) \rangle = \frac{-1}{\pi N} \operatorname{Im} \operatorname{tr} G(\mu + i0). \tag{2.1}$$

Average density of states is the quantity I am going to calculate. Averaging over GUE is produced in the following way.

$$\begin{aligned} \tilde{D}^{\alpha\beta}_{\mu\nu} \equiv \left\langle \varphi_{\alpha\beta} \varphi^*_{\mu\nu} \right\rangle &= \int D\varphi \, P(\varphi) \varphi_{\alpha\beta} \varphi^*_{\mu\nu} \qquad (2.2) \\ &= \int \prod_{i=1}^{N} \frac{d\varphi_{ii}}{\sqrt{2\pi/Nm^2}} \prod_{i<j}^{N} \frac{d \operatorname{Re} \varphi_{ij} \, d \operatorname{Im} \varphi_{ij}}{\pi/Nm^2} \varphi_{\alpha\beta} \varphi^*_{\mu\nu} \exp\left[-N \frac{m^2}{2} \sum_{i,j=1}^{N} |\varphi_{ij}|^2 \right] \\ &= \frac{1}{Nm^2} \delta_{\alpha\mu} \delta_{\beta\nu}. \end{aligned}$$

Similarly, when time dependence is present, the similar logic works in the Fourier space.

$$\varphi(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \varphi_\omega e^{-i\omega t} = \frac{1}{T} \sum_{\omega=-\infty}^{\infty} \varphi_\omega e^{-i\omega t}.$$

Since φ is Hermitian, its Fourier components satisfy $\varphi^+_\omega = \varphi_{-\omega}$. Pair correlation function is given by

$$\left\langle \varphi^{\omega}_{\alpha\beta} \bar{\varphi}^{\omega'}_{\mu\nu} \right\rangle = \frac{T \delta_{\omega,\omega'}}{N} \frac{\delta_{\alpha\mu} \delta_{\beta\nu}}{\omega^2 + m^2}.$$

In real space it translates to

$$\begin{aligned} D^{\alpha\beta}_{\mu\nu}(t) \equiv \langle \varphi_{\alpha\beta}(t) \bar{\varphi}_{\mu\nu}(0) \rangle &= \frac{\delta_{\alpha\mu}\delta_{\beta\nu}}{N} \int \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{\omega^2 + m^2} = \frac{\delta_{\alpha\mu}\delta_{\beta\nu}}{2Nm} e^{-m|t|} \\ &= \frac{\sigma^2(t)}{N} \delta_{\alpha\mu} \delta_{\beta\nu}. \end{aligned}$$

Here variance σ^2 is different for Please note that $D^{\alpha\beta}_{\mu\nu}(0) \neq \tilde{D}^{\alpha\beta}_{\mu\nu}$ from (2.2).

Next I'm interested in developing perturbation theory in $N \gg 1$. Let me at first formally expand Green's function in φ and estimate which diagrams give leading order contribution.

$$G(z) = \frac{1}{z} \sum_{k=0}^{\infty} \left\langle \frac{\varphi^k}{z^k} \right\rangle.$$

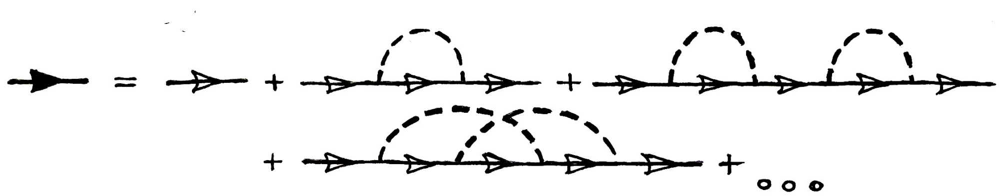

Figure 1: Several simplest contributions to the Green's function. Solid line — bare Green's function $G^0 = 1/z$, dashed line — impurity interaction line D. Here I use single dashed lines instead of double lines for simplicity.

Obviously, only even powers give non–zero contributions. It is useful to introduce bare Green's function $G_0(z) = \frac{1}{z}$, which proportional to identity matrix. In such terms,

$$G = G_0 + G_0 \langle \varphi G_0 \varphi \rangle G_0 + G_0 \langle \varphi G_0 \varphi G_0 \varphi G_0 \varphi \rangle G_0 + \dots$$

It is custom to define self–energy according to

$$G = \frac{1}{z - \Sigma} \quad \Leftrightarrow \quad G = G_0 + G_0 \Sigma G.$$

By definition self–energy only contains irreducible contributions.

$$z\Sigma_{ij}^{(1)} = D_{jk}^{ik} = \frac{\delta_{ij}}{2m},$$
$$z^3\Sigma_{ij}^{(2)} = D_{j\beta}^{i\alpha} D_{\beta\gamma}^{\alpha\gamma} + D_{\beta\gamma}^{i\alpha} D_{j\beta}^{\alpha\gamma} = \frac{\delta_{ij}}{(2m)^2}\left[1 + \frac{1}{N^2}\right].$$

On the language of double–lines perturbation theory diagrams it looks like

Next step is to recognize that some contributions are negligible in the $N \to \infty$ limit. Every impurity line (dashed double line) contains $1/N$ factor, while each loop gives a factor of N, that allows to understand what leading contributions look like. Such contributions are given by so-called self–crossing diagrams, because interaction lines on them cross when draw in the same half–plane. Once only the leading terms are left, all the contributions could be taken into account via single self–consistent equation, so–called self–consistent Born approximation (SCBA).

$$\Sigma_{ij} = \langle \varphi G \varphi \rangle_{ij} = D_{j\beta}^{i\alpha} G_{\alpha\beta} = \frac{\delta_{ij}}{2Nm} \operatorname{tr} G.$$

Let me define $g = \frac{1}{N} \operatorname{tr} G$, which is simply a number. As a consequence of translation invariance $G_{ij} = g\delta_{ij}$. SCBA equation allows to find

$$g = \frac{1}{z} + \frac{1}{z}\frac{1}{2m} g^2 \quad \Leftarrow \quad g(z) = mz - \sqrt{(mz)^2 - 2m}.$$

I have chosen the solution that gives positive density of states according to (2.1).

$$\rho(z) = \frac{-1}{\pi} \operatorname{Im} g(z + i0) = \frac{2m}{\pi}\sqrt{\frac{1}{2m} - \frac{z^2}{4}}.$$

In order to get answer for time–independent case, one should just substitute $2m \mapsto m^2$.

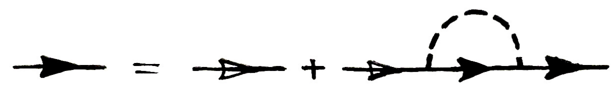

Figure 2: Graphical representation of the SCBA equation. Here I use single dashed lines instead of double lines for simplicity.

2.2 Density–density correlations

Now, I'm interested in calculating correlations of density of states. I introduce two–point Green's functions and express DOS–DOS correlations through them.

$$\begin{aligned} G^{\alpha\beta}_{\mu\nu}(z,w,t) &= \left\langle \left(\frac{1}{z-\varphi(t)}\right)_{\alpha\beta} \left(\frac{1}{w-\varphi(0)}\right)_{\mu\nu} \right\rangle, \\ \rho(\mu,\nu,t) &= \left\langle \frac{1}{N}\operatorname{tr}\delta(\mu-\varphi(t))\frac{1}{N}\operatorname{tr}\delta(\nu-\varphi(0)) \right\rangle \\ &= \frac{-1}{4\pi^2}\left[g(++)+g(--)-g(+-)-g(-+)\right]. \end{aligned} \tag{2.3}$$

Here $G(\pm,\pm')$ is a shorthand notation for $N^{-2}G^{\alpha\alpha}_{\beta\beta}(\mu\pm i0,\nu\pm' i0)$.Finally, I am only interested in connected part $\rho_c(\mu,\nu)=\rho(\mu,\nu)-\rho(\mu)\rho(\nu)$, hence a connected two–point Green's function.

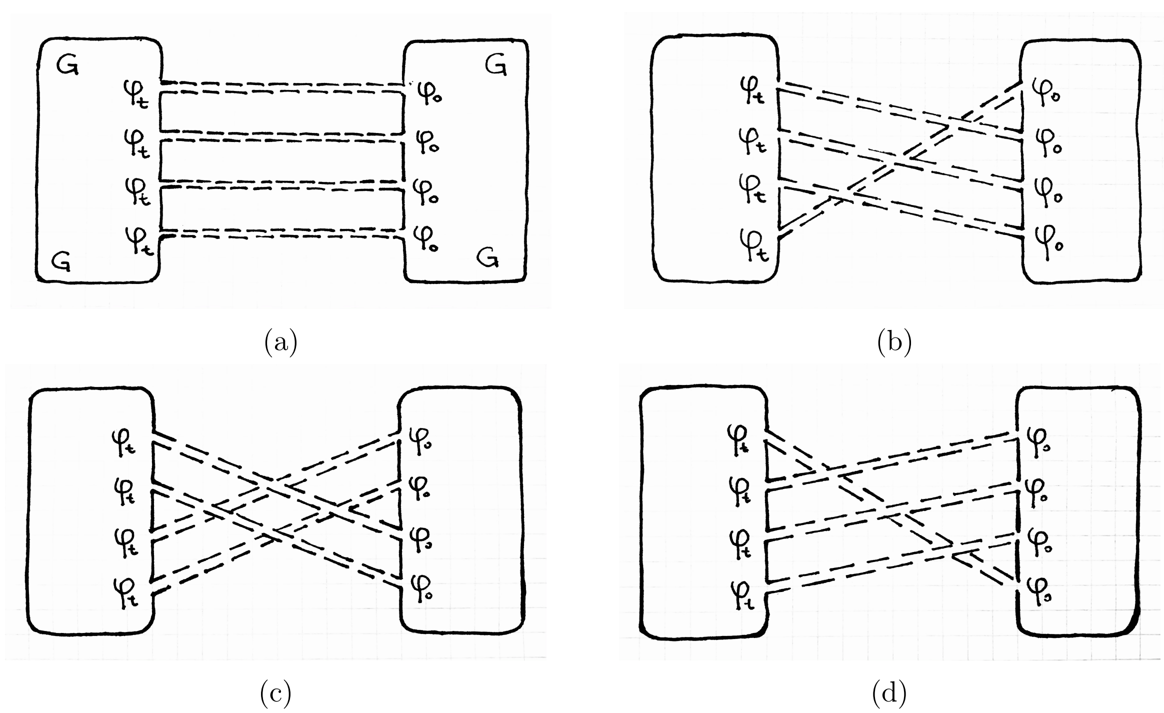

Figure 3: Four different ways to contract $\langle \operatorname{tr}\varphi^4(t)\operatorname{tr}\varphi^4(0)\rangle$. Each coupling is determined by how I connect say first $\varphi(t)$ in the left bubble to some particular $\varphi(0)$ on the right. Other ways of contraction (not present here) have sub–leading order in $1/N$.

$$g_c(z,w,t) = \left\langle \frac{1}{N}\operatorname{tr}\frac{1}{z-\varphi(t)}\frac{1}{N}\operatorname{tr}\frac{1}{w-\varphi(0)} \right\rangle_c \tag{2.4}$$
$$= \frac{1}{N^2}\frac{1}{zw}\sum_{m,n=0}^{\infty}\left\langle \operatorname{tr}\left(\frac{\varphi(t)}{z}\right)^m \operatorname{tr}\left(\frac{\varphi(0)}{w}\right)^n \right\rangle_c$$

Leading order contributions to g_c are given by ladder diagrams and vertex corrections with all bare Green's functions replaced by SCBA values G. First I calculate ladder diagrams that connect different bubbles: they consist of contributions with $m = n$ and $\varphi^m(t)$ connected to $\varphi^n(0)$ with parallel lines. Once both loops are expanded to n–th order there are n ways to connect one loop to another. Once I have connected some $\varphi(t)$ to any of the $\varphi(0)$, the diagram is fixed, there are $m = k+1$ ways to do it.

$$N^2 g_c^{(\text{ladder})}(z,w,t) = g_z^2 g_w^2 \sigma^2(t)\sum_{k=0}^{\infty}(k+1)\left[g_z g_w \sigma^2(t)\right]^k = \frac{g_z^2 g_w^2 \sigma^2(t)}{\left[1 - g_z^2 g_w^2 \sigma^4(t)\right]^2}$$

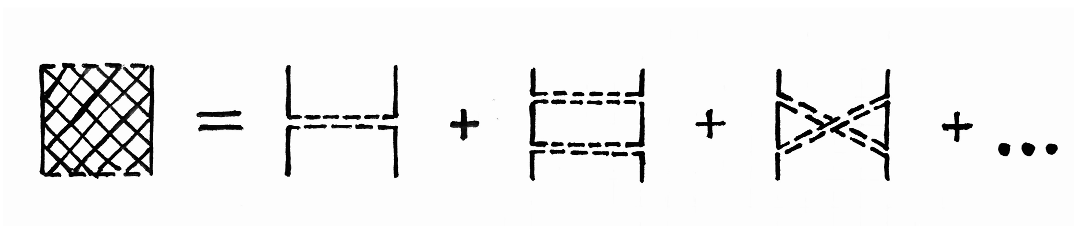

Figure 4: Summation of all the possible contraction from different traces in $\langle \operatorname{tr}\varphi^n(t)\operatorname{tr}\varphi^n(0)\rangle$.

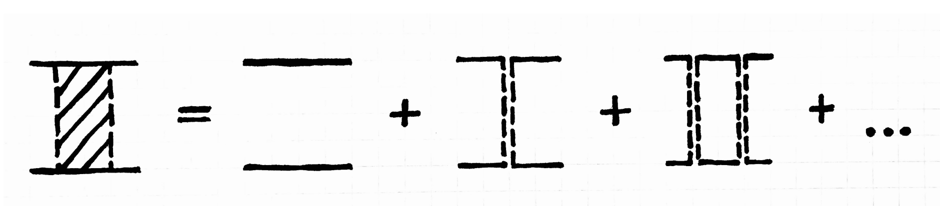

Figure 5: Vertex renormalization.

Now I'm taking into account so-called vertex renormalizations. These diagrams also have a ladder inserted into the same bubble. Altogether in $N \to \infty$ limit I have

$$N^2 g_c(z,w,t) = \frac{1}{1-\sigma^2(0)g_z^2}\frac{g_z^2 g_w^2 \sigma^2(t)}{\left[1 - g_z g_w \sigma^2(t)\right]^2}\frac{1}{1-\sigma^2(0)g_w^2}. \tag{2.5}$$

Please note a misprint in [1, (2.10)], there is an extra square. Let me rewrite the result in a simpler form by introducing new variables. I name $\sigma^2(t) = \sigma_0^2 e^{-m|t|}$ and define angles

$$\sin\theta = \frac{z}{2\sigma_0}, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \quad \text{when} \quad -2\sigma_0 < z < 2\sigma_0,$$
$$\sin\phi = \frac{w}{2\sigma_0}, \qquad -\frac{\pi}{2} < \phi < \frac{\pi}{2}, \quad \text{when} \quad -2\sigma_0 < w < 2\sigma_0.$$

In these terms

$$g(z \pm i0) = \frac{1}{\sigma_0^2}\left[\frac{z}{2} \mp i\sqrt{\sigma_0^2 - \left(\frac{z}{2}\right)^2}\right] = \frac{\mp i}{\sigma_0}e^{\pm i\theta}, \qquad g^2(z \pm i0) = -\frac{e^{\pm 2i\theta}}{\sigma_0^2}.$$

$$g(w \pm i0) = \frac{1}{\sigma_0^2}\left[\frac{w}{2} \mp i\sqrt{\sigma_0^2 - \left(\frac{w}{2}\right)^2}\right] = \frac{\mp i}{\sigma_0}e^{\pm i\phi}, \qquad g^2(w \pm i0) = -\frac{e^{\pm 2i\phi}}{\sigma_0^2}.$$

Which allows for a shorter expression

$$N^2 g_c(\pm, \pm', t) = \frac{1}{8\sigma_0^2}\frac{1}{\cos\theta\cos\phi} \cdot \frac{1}{(\pm\pm') + \cos\left[im|t| \pm \theta \pm' \phi\right]}.$$

Finally, I use expression (2.3) to find connected part of the density–density correlation function.

$$\begin{aligned} N^2\rho_c(z,w,t) = \frac{-1}{4\pi^2}\frac{1}{8\sigma_0^2}\frac{1}{\cos\theta\cos\phi}\Bigg[&\frac{1}{1+\cosh m|t|\cos(\theta+\phi) - i\sinh m|t|\sin(\theta+\phi)} + \\ &+ \frac{1}{1+\cosh m|t|\cos(\theta+\phi) + i\sinh m|t|\sin(\theta+\phi)} - \\ &- \frac{1}{-1+\cosh m|t|\cos(\theta-\phi) - i\sinh m|t|\sin(\theta-\phi)} - \\ &- \frac{1}{-1+\cosh m|t|\cos(\theta-\phi) + i\sinh m|t|\sin(\theta-\phi)}\Bigg] \\ = \frac{-1}{16\pi^2\sigma_0^2}\frac{1}{\cos\theta\cos\phi}\left[\frac{1+\cosh m|t|\cos(\theta+\phi)}{(\cosh m|t| + \cos(\theta+\phi))^2} + \frac{1-\cosh m|t|\cos(\theta-\phi)}{(\cosh m|t| - \cos(\theta-\phi))^2}\right]& \end{aligned}$$

That is the final answer (1.2). When $z = w$ energies coincide finite time smoothens the divergence

$$N^2\rho_c(z,z,t) = \frac{-1}{16\pi^2\sigma_0^2}\frac{1}{\cos^2\theta}\left[\frac{1+\cosh m|t|\cos 2\theta}{(\cosh m|t| + \cos 2\theta)^2} + \frac{1}{1-\cosh m|t|}\right].$$

In the limit of large times correlations decay exponentially

$$N^2\rho_c(z,w,t) = \frac{\tan\theta\tan\phi}{4\pi^2\sigma_0^2}e^{-m|t|}, \quad t \to \infty.$$

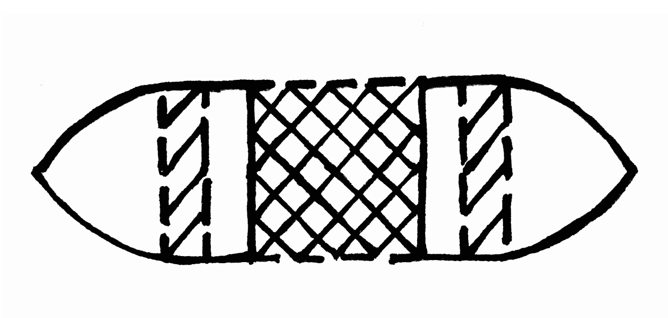

Figure 6: Diagram for the g_c in large size limit.

2.2.1 Time–independent case

Here for completeness I provide the answer for the case of time–independent ensemble. When

$$P(\varphi) \propto \exp\left[-\frac{N}{2}m^2\varphi^2\right], \qquad g(z) = \frac{m^2}{2}\left[z - \sqrt{z^2 - \frac{4}{m^2}}\right]$$

eigenstates density has the same form as before but with $\sigma_0^2 = 1/m^2$.

$$\rho(z)dz = \frac{m^2}{\pi}\sqrt{\frac{1}{m^2} - \frac{z^2}{4}} = \frac{2}{\pi}\cos^2\theta d\theta, \qquad z = \frac{2}{m}\sin\theta.$$

Connected Green's function in $N \to \infty$ limit reads

$$N^2 g_c(z,w,t) = \frac{1}{1 - g_z^2/m^2}\frac{g_z^2 g_w^2/m^2}{[1 - g_z g_w/m^2]^2}\frac{1}{1 - g_w^2/m^2}.$$

Similarly, connected part of density–density correlations is

$$N^2 \rho_c(z,w,t) = \frac{-m^2}{16\pi^2}\frac{1}{\cos\theta\cos\phi}\left[\frac{1}{1+\cos(\theta+\phi)} + \frac{1}{1-\cos(\theta-\phi)}\right].$$

2.2.2 Alternative derivation

Here I briefly repeat an alternative derivation of (1.2) given in [2]. Connected Green's function from (2.4) could be presented as

$$\begin{aligned} N^2 g_c(z,w,t) &= \partial_z\partial_w \left\langle \operatorname{tr}\ln(z - \varphi(t)) \operatorname{tr}\ln(w - \varphi(0))\right\rangle_c \\ &= \partial_z\partial_w \left\langle \operatorname{tr}\ln\left(1 - \frac{\varphi(t)}{z}\right) \operatorname{tr}\ln\left(1 - \frac{\varphi(0)}{w}\right)\right\rangle_c \\ &= \partial_z\partial_w \sum_{n,m=1}^{\infty} \frac{1}{nm}\frac{\langle \operatorname{tr}\varphi^n(t) \operatorname{tr}\varphi^m(0)\rangle_c}{z^n w^m} \end{aligned} \tag{2.6}$$

In the $N \to \infty$ limit main diagrams correspond to elements with $n = m$ where $\operatorname{tr}\varphi^n(t)$ is connected to $\operatorname{tr}\varphi^m(0)$ with parallel impurity lines. By parallel I mean that if each trace is a ring of solid line, drawn one inside the other, then dashes lines do not intersect. There are $m = n$ ways to connect lines in such way. Hence

$$\begin{aligned} N^2 g_c(z,w,t) &= \partial_z\partial_w \sum_{n,m=1}^{\infty} \frac{\delta_{nm}}{nm}\frac{m}{z^n w^m}\sigma^{2m}(t) \\ &= -\partial_z\partial_w \ln\left(1 - \frac{\sigma^2(t)}{zw}\right). \end{aligned}$$

All that I am missing is contributions inside the same trace which lead to renormalization of bare Green's functions to SCBA values $\frac{1}{z} \mapsto g(z)$. Finally,[2]

$$\begin{aligned} N^2 g_c(z,w,t) &= -\partial_z\partial_w \ln\left(1 - \sigma^2(t) g_z g_w\right) \\ &= \frac{g_z' g_w' \sigma^2(t)}{[1 - \sigma^2(t) g_z g_w]^2} \end{aligned}$$

[2] I note that there is a missing minus sign in [2, (22)].

In order to see that the present result coincides with (2.5) I need an expression for the derivative of SCBA Green's function which follows from SCBA equation.

$$z = \frac{1}{g} + \sigma_0^2 g \qquad \Rightarrow \qquad g'_z = \frac{-g^2}{1 - \sigma_0^2 g^2}.$$

In order to see how two derivation correspond to one another, one should note that differentiation of a bare Green's function $\partial g_0(z) = \partial \frac{1}{z} = -\frac{1}{z^2} = -g_0^2(z)$ is proportional to square of Green's function. Arbitrary diagram from sum (2.6) with dressed Green's function should be presented as the sum SCBA Green's functions and then differentiated. It is also clear why such approach does not require separate vertex correction, which is needed in the Berezin–Zee calculation [1].

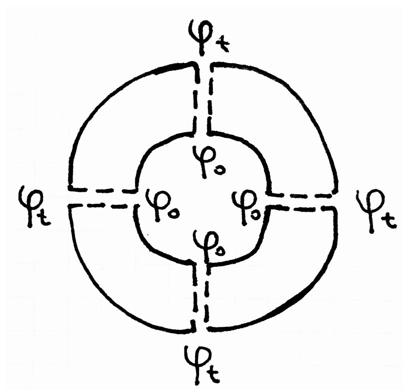

Figure 7: Different way to contract $\langle \operatorname{tr} \varphi^4(t) \operatorname{tr} \varphi^4(0) \rangle$ follow from that picture by rotating one of the ring with respect to the other. Diagrams on Figure 3 are produced by differentiating one of the Green's function in each ring.

3 Complex Wishart Ensemble

In literature one may also encounter another way to introduce Gaussian random matrix ensemble. In contrast to unitary, orthogonal and symplectic ensembles, there is also so–called Complex Wishart Ensemble (CWE). The random matrix (Hamiltonian) is the product of two not necessary square, not Hermitian (just complex) matricies $\varphi = \alpha^+\alpha$.

$$\langle \ldots \rangle = \int \prod_{i,j=1}^{N} \frac{d\alpha_{ij} \wedge d\alpha_{ij}^*}{2\pi i} \ldots \exp\left(-\frac{N}{\sigma_0^2} \sum_{i,j=1}^{N} |\alpha_{ij}|^2 \right), \qquad \frac{dz \wedge dz^*}{2\pi i} \equiv \frac{d\,\mathrm{Re}\, z\, d\,\mathrm{Im}\, z}{\pi}.$$

Pair correlator in that case reads

$$\left\langle \alpha_{ij}^* \alpha_{kl} \right\rangle = \langle \alpha_{ij} \alpha_{kl}^* \rangle = \frac{\sigma_0^2}{N} \delta_{ik} \delta_{jl}.$$

3.1 Density of states

I start by deriving the density of eigenstates (per node) in large matrix size limit.

$$\rho(\mu) \equiv \left\langle \delta(\mu - \alpha^+\alpha) \right\rangle = \frac{-1}{\pi N} \mathrm{Im}\,\mathrm{tr}\, G(\mu + i0).$$

Green's function G should be expanded, then analyse which terms produce the largest contribution.

$$G = (G_0^{-1} - \varphi)^{-1} = G_0 + G_0 \varphi G_0 + G_0 \varphi G_0 \varphi G_0 + \ldots, \qquad \varphi = \alpha^+\alpha.$$

On Figure 8 I present all the contributions up to φ^3.

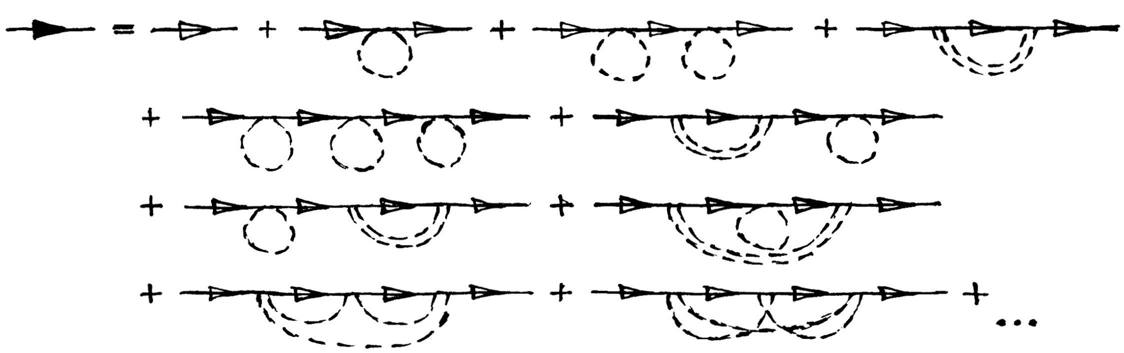

Figure 8: First three order of Green's function's expansion. Here I use single dashed lines instead of double lines for simplicity.

Below I analyse all the third order terms, the diagrams have the following analytic expressions.

$$\begin{aligned}
\left\langle G_0 \alpha^+ \alpha G_0 \alpha^+ \alpha G_0 \alpha^+ \alpha G_0 \right\rangle &= G_0 \langle \alpha^+ \alpha \rangle G_0 \langle \alpha^+ \alpha \rangle G_0 \langle \alpha^+ \alpha \rangle G_0 && \propto N \cdot N \cdot N \cdot G_0^4 \\
&+ G_0 \langle \alpha^+ \langle \alpha G_0 \alpha^+ \rangle \alpha \rangle G_0 \langle \alpha^+ \alpha \rangle G_0 && + N \cdot \mathrm{tr}\, G_0 \cdot N \cdot G_0^3 \\
&+ G_0 \langle \alpha^+ \alpha \rangle G_0 \langle \alpha^+ \langle \alpha G_0 \alpha^+ \rangle \alpha \rangle G_0 && + N \cdot \mathrm{tr}\, G_0 \cdot N \cdot G_0^3 \\
&+ G_0 \langle \alpha^+ \langle \alpha G_0 \langle \alpha^+ \alpha \rangle G_0 \alpha^+ \rangle \alpha \rangle G_0 && + N \cdot \mathrm{tr}\, G_0^2 \cdot N \cdot G_0^2 \\
&+ G_0 \langle \alpha^+ \langle \alpha G_0 \alpha^+ \rangle \langle \alpha G_0 \alpha^+ \rangle \alpha \rangle G_0 && + N \cdot \mathrm{tr}\, G_0 \cdot \mathrm{tr}\, G_0 \cdot G_0^2 \\
&+ G_0 \alpha^+ \alpha G_0 \alpha^+ \alpha G_0 \alpha^+ \alpha G_0 && + G_0^4.
\end{aligned}$$

Out of $3! = 6$ possible contractions only 5 contribution have the same (largest) order in $NG_0 = N/\varepsilon$, while one term is small — it could be thrown away. That allows to write down Self–consistent Born approximation (Fig. 9).

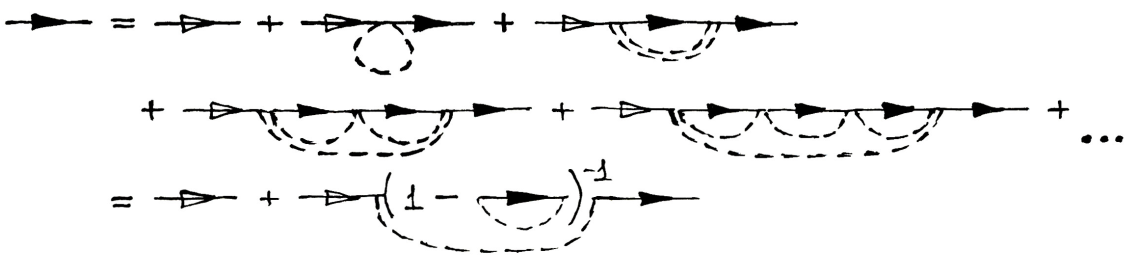

Figure 9: Self–consistent Born (SCBA) equation for Green's function G. Here I use single dashed lines instead of double lines for simplicity.

Diagrams structure is such that self–energy could not be simply expressed through the Green's function as usual, but the whole series of Green's functions and impurity lines is required instead. That is why it is convenient to introduce new entity f – number that is contained inside the self–energy (under the impurity line rainbow).

$$G = G_0 + G_0 \cdot \sigma_0^2 f \cdot G.$$

Within SCBA, number f is given the series presented on Figures 9 and 10.

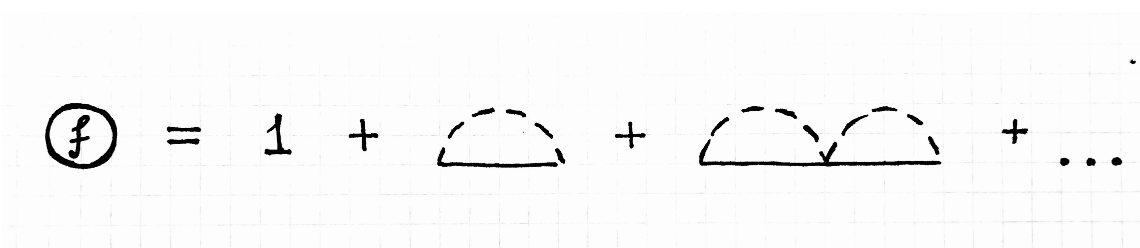

Figure 10: Graphical representation of number f.

$$f = 1 + \frac{\sigma_0^2}{N} \operatorname{tr} G + \frac{\sigma_0^2}{N} \operatorname{tr} G \cdot \frac{\sigma_0^2}{N} \operatorname{tr} G + \dots = \frac{1}{1 - \frac{\sigma_0^2}{N} \operatorname{tr} G} = \frac{1}{1 - \sigma_0^2 g}.$$

Apparently matrix G is diagonal $G_{ij} = g\delta_{ij}$, so I introduce $G_0^{ij} = g_0 \delta^{ij}$, which allows to write equation on g.

$$g = g_0 + g_0 \cdot \sigma_0^2 f \cdot g = g_0 + g_0 \frac{\sigma_0^2}{1 - \sigma_0^2 g} g.$$

Retarded solution on SCBA Green's function is

$$g(\varepsilon) = \frac{1}{2\sigma_0^2} \left(1 - i\sqrt{\frac{4\sigma_0^2}{\varepsilon} - 1} \right).$$

That finally allows to find density of states (per node).

$$\rho(\varepsilon) = \frac{-1}{\pi N} \operatorname{Im} \operatorname{tr} G = \frac{\theta(\varepsilon)\theta(4\sigma_0^2 - \varepsilon)}{2\pi\sigma_0^2} \sqrt{\frac{4\sigma_0^2}{\varepsilon} - 1}.$$

I note that total number of states is correct $\int \rho(\varepsilon) d\varepsilon = 1$. It is clear that there are no states for negative energies by design. However the absence of eigenvalues with $\varepsilon > 4\sigma_0^2$ is a byproduct of large N approximation. Actually, there is an exponentially small (in N) number states for $\varepsilon > 4\sigma_0^2$.

3.2 Density–density correlations

Here I calculate the correlations of density for CWE in analogy for what was done in the previous section for GUE. That calculation is taken from [2]. Now that all the preliminary work is done, it is quite straightforward to generalize the result on the case of CWE.

Density–density correlations are most easily computed through the $\partial - -\log$ formula.

$$\begin{aligned} N^2 g_c(z, w) &= \partial_z \partial_w \left\langle \operatorname{tr} \ln(z - \alpha^+\alpha) \operatorname{tr} \ln(w - \alpha^+\alpha) \right\rangle_c \\ &= \partial_z \partial_w \sum_{n,m=1}^{\infty} \frac{1}{nm} \frac{\langle \operatorname{tr}(\alpha^+\alpha)^n \operatorname{tr}(\alpha^+\alpha)^m \rangle_c}{z^n w^m}. \end{aligned}$$

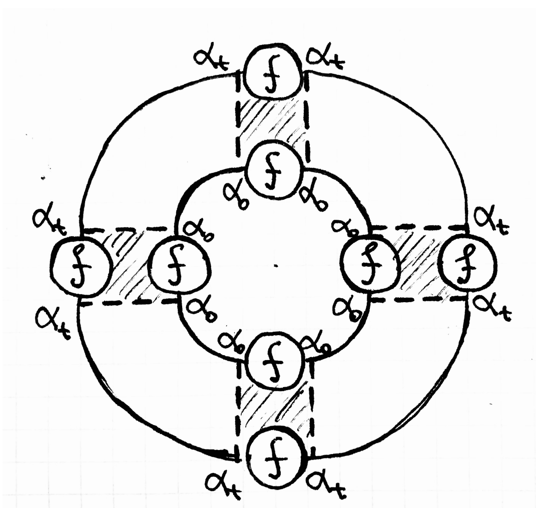

Figure 11: Feynman diagram for $\langle \operatorname{tr}(\alpha^+\alpha)^4 \operatorname{tr}(\alpha^+\alpha)^4 \rangle_c$.

Similar to the GUE case there are n ways to couple $\langle \operatorname{tr}(\alpha^+\alpha)^n \operatorname{tr}(\alpha^+\alpha)^n \rangle_c$. Contractions within the same trace renormalize bare Green's function but also renormalize the parallel double impurity lines inside by multiplying each on f_z and f_w respectively.

$$\begin{aligned} N^2 g_c(z, w) &= -\partial_z \partial_w \ln \left(1 - \sigma_0^4 \frac{g_z}{1 - \sigma_0^2 g_z} \frac{g_w}{1 - \sigma_0^2 g_w} \right) \\ &= \frac{\sigma_0^4 g_z' g_w'}{(1 - \sigma_0^2 g_z - \sigma_0^2 g_w)^2} \end{aligned}$$

In order to simplify that expression I compute the derivative implicitly using SCBA equation.

$$z = \frac{1}{g} + \frac{\sigma_0^2}{1 - \sigma_0^2 g} = \frac{1}{g(1 - \sigma_0^2 g)} \quad \Rightarrow \quad g_z' = -\frac{1 - 2\sigma_0^2 g}{g^2(1 - \sigma_0^2 g)^2}.$$

4 Conclusion

I have reproduced calculation of density–density correlation function for the Gaussian Unitary Ensemble [1] and Complex Wishart Ensemble [2].

References

[1] E. Brézin and A. Zee. Correlation functions in disordered systems. *Phys. Rev. E*, 49:2588–2596, Apr 1994.

[2] J. Jurkiewicz, G Lukaszewski, and M.A. Nowak. Diagrammatic approach to fluctuations in the wishart ensemble. *Acta Phys. Pol. B*, 39:799, 2008.