The Susquehanna Problem Solving Assessment

Duration: 1 hour

Test had 17 questions, but you're not expected to solve all of it.
Here I reproduce some of the questions. Solutions may contain mistakes.

Problem 1. Factory A makes 40% red widgets and 60% black widgets. Factory B makes 80% red widgets and 20% black widgets. Two widgets are sampled uniformly at random from one of the companies, also selected uniformly at random. The widgets are both red. Compute the probability they were from Factory A.

Answer 1. 1/17.

Solution 1. If Factory A was chosen $\mathbb{P}\left[\text{red}, \text{red}|A\right] = 0.4^2$, similarly $\mathbb{P}\left[\text{red}, \text{red}|B\right] = 0.8^2$.

$$\mathbb{P}\left[A|\text{red}, \text{red}\right] = \frac{\mathbb{P}\left[\text{red}, \text{red}|A\right]}{\mathbb{P}\left[\text{red}, \text{red}|A\right] + \mathbb{P}\left[\text{red}, \text{red}|B\right]} = \frac{4^2}{4^2+8^2} = \frac{1}{1+4^2} = \frac{1}{17}.$$

Problem 2. You roll a fair 6-sided die two times and get paid the higher of the two rolls in dollars if the rolls are different. If they are the same, you get paid \$0. Compute your expected payoff from this game.

Answer 2. \$3.89.

Solution 2. All possible payouts can be represented with symmetric matrix where each entry is equally likely.

$$\begin{pmatrix} 0 & 2 & 3 & 4 & 5 & 6 \\ & 0 & 3 & 4 & 5 & 6 \\ & & 0 & 4 & 5 & 6 \\ & & & 0 & 5 & 6 \\ & & & & 0 & 6 \\ & & & & & 0 \end{pmatrix}$$

So we need to sum up all the elements and multiply them by $\frac{1}{36}$, since matrix is symmetric it's enough to sum $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 + 5 \cdot 6 = 70$ and divide it by 18. As a sanity check the answer should be somewhere between \$1 and \$6, but probably closer to the larger end.

Let's see if we can extrapolate this answer to the case of n–face die. Our expected payout is

$$\begin{aligned} P &= \frac{2}{n^2} \sum_{i<j}^{1\ldots n} \max(i,j) = \frac{2}{n^2} \sum_{i=1}^{n} \sum_{j=i}^{n} j \\ &= \frac{2}{n^2} \sum_{i=1}^{n} \left[\frac{n(n+1)}{2} - \frac{i(i+1)}{2} \right] \\ &= (n+1) - \frac{1}{n} \frac{(n+1)(n+\frac{1}{2})}{3} - \frac{1}{n} \frac{(n+1)}{2} \\ &= \frac{2}{3} \left(n - \frac{1}{n} \right). \end{aligned}$$

Problem 3. Assume that in a bakery, each customer buys only one item at a time. There is a 70% chance a customer will buy a croissant and a 30% chance a customer will buy a muffin. There are only 2 muffins left and 5 people are still waiting in line. Compute the probability that these two muffins will be sufficient, i.e. no customer will want a muffin and find that there are none left.

Answer 3. 84 %.

Solution 3. Let's compute the probabilities of all the possible ways for no more then two muffins requests. Let $p = 0.7$.

$$P = p^5 + \binom{5}{1} p^4(1-p) + \binom{5}{2} p^3(1-p)^2$$
$$= 0.7^2 + 5 \cdot 0.7^4 \cdot 0.3 + 10 \cdot 0.7^3 \cdot 0.3^2$$

Problem 4. You baked 6 indistinguishable snickerdoodle cookies and 8 indistinguishable chocolate chip cookies. Compute the number of ways to arrange 7 of these cookies into a straight line.

Answer 4. 127.

Solution 4. Let's consider arrangements that have fixed number $0 \leqslant k \leqslant 6$ of snickerdoodle coockies in the line. Next let's sum over all possible k.

$$\sum_{k=0}^{6} \binom{7}{k} = \sum_{k=0}^{7} \binom{7}{k} - 1 = (1+1)^7 - 1 = 127.$$

Remark. I made a mistake in this problem during the test.

Problem 5. Three cats are competing in a jumping contest. The most athletic cat wins with probability $\frac{3}{4}$, the least athletic cat wins with probability $\frac{1}{16}$, and the remaining cat wins with probability $\frac{3}{16}$. I picked a cat uniformly at random to cheer on but it did not win. Compute the probability I picked the most athletic cat.

Answer 5. 1/8.

Solution 5. Let's call most athletic cat A, least athletic C and average cat B. Since the cat did not win

$$P(A|\text{lost}) = \frac{P(\text{lost}|A)}{P(\text{lost}|A) + P(\text{lost}|B) + P(\text{lost}|C)} = \frac{4}{15+13+4} = \frac{1}{8}.$$

Problem 6. A spinner has three regions, and the probabilities of landing in each region are $\frac{1}{4}$, $\frac{1}{4}$, and $\frac{1}{2}$. Compute the expected number of spins it would take to land in two distinct regions.

Answer 6. 8/3.

Solution 6. Let's label regions A, B, and C. Probabilities of landing on each regions are $p_A = p_B = \frac{1}{4}$ and $p_C = \frac{1}{2}$. Let x_A be the random variable defined as follows. We just landed on region A on the last spin and we a going to spin until we land on two distinct regions in a row, the number of spins required is the value of x_A. If we landed on B immediately, then $x_A = 1$. If we got AAC, then $x_A = 3$, etc. Similarly we define x_B and x_C. Finally x is the random variable equal to the number of spins until we land on two distinct regions in a row (assuming that there were no prior spins). We are asked to find $X = \mathbb{E}[x]$.

$$\begin{aligned}
X &= p_A(1+X_A) + p_B(1+X_B) + p_C(1+X_C), \\
&= 1 + p_A X_A + p_B X_B + p_C X_C, \\
X_A &= (1-p_A) + p_A(1+X_A), \\
X_B &= (1-p_B) + p_B(1+X_B), \\
X_C &= (1-p_C) + p_C(1+X_C)
\end{aligned}$$

We can immediately find $X_A = \frac{1}{1-p_A} = \frac{4}{3}$, $X_B = \frac{1}{1-p_B} = \frac{4}{3}$, and $X_C = \frac{1}{1-p_C} = 2$. From here

$$X = 1 + \sum_i^{A,B,C} \frac{p_i}{1-p_i} = -2 + \sum_i^{A,B,C} \frac{1}{1-p_i} = \frac{8}{3}$$

Problem 7. A gardener is eagerly waiting for his two favorite flowers to bloom. The purple flower will blossom at some point uniformly at random in the next 20 days and be in bloom for exactly 4 days. Independent of the purple flower, the red flower will blossom at some point uniformly at random in the next 20 days and be in bloom for exactly 8 days. Compute the probability that both flowers will simultaneously be in bloom at some point in time.

Answer 7. 93/200.

Solution 7. This is a poorly formulated question. So I am going to interpret it in a some way.

Let x be the first day of bloom of purple flower, and y the frist day of red flower. Let's enumerate today as 0 and then next 20 are numbered 1, 2, ..., 20. Then $1 \leqslant x, y \leqslant 20$. If purple flower is in bloom for 4 days it means it's in bloom on days x, $x+1$, $x+2$, and $x+3$. Note that it only starts blossoming on days 1, 2, ..., 20, but it continue to blossom on days 21, 22, ...

$$P = \frac{1}{20^2} \sum_{x=1}^{20} \sum_{y=1}^{20} \theta\left([x, x+3] \cap [y, y+7]\right).$$

Let's find the probability of the opposite event happening: both flower are in bloom on non–overlapping days.

$$1 - P = P_{x+3<y} + P_{y+7<x},$$
$$P_{x+3<y} = \frac{1}{20^2} \sum_{x=1}^{16} \sum_{y=x+4}^{20} 1,$$
$$P_{y+7<x} = \frac{1}{20^2} \sum_{y=1}^{12} \sum_{x=y+8}^{20} 1.$$

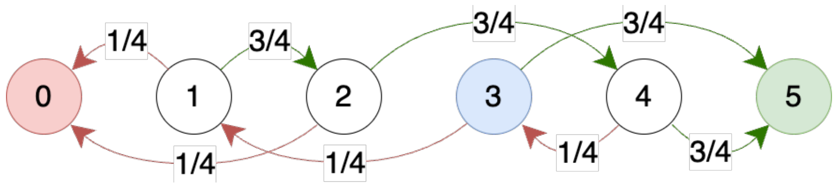
It's not hard to compute $P_{x+3<y} = \frac{17}{50}$, and $P_{y+7<x} = \frac{39}{200}$.

Problem 8. A deck contains eight cards: two 10s, two Js, two Qs, two Ks. You are dealt five cards without replacement from this deck. Compute the expected number of pairs in your hand.

Answer 8. 10/7.

Solution 8. Overall there are 8 choose 5 = 56 combinations. Since we have only 4 types of card, we would always have at least one pair due to the pigeonhole principle. Let's conside 3 cards that are not dealt. If there is a pair among these 3 cards it looks like XXY, where there 4 ways to choose X and 6 ways to choose Y, which means 24 combinations correspond to having 1 pair among undealt card, i.e. having 2 pairs among 5 dealt card, the rest $56 - 24 = 32$ cases correspond to having a signle pair among 5 dealt card, and zero pairs among undealt card.

Problem 9. Suppose you have 3 tokens for a betting game and your goal is to bring your fortune up to 5 tokens before running out of tokens. You devised a bold betting strategy where every turn, you will bet as many tokens as possible against the house but not any more than necessary to bring you to 5 tokens. You always have a probability of $\frac{3}{4}$ to win the bet. Compute the probability you reach 5 tokens before running out.



Markov chain.

Answer 9. $\frac{219}{247} \approx 89\%$.

Solution 9. We start by represeting strategy as a Markov chain. Transition matrix

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & \frac{3}{4} & 0 & 0 \\ \frac{1}{4} & 0 & 0 & \frac{3}{4} & 0 \\ 0 & \frac{1}{4} & 0 & 0 & \frac{3}{4} \\ 0 & 0 & \frac{1}{4} & 0 & \frac{3}{4} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Let's denote probability of reaching state with 5 token starting from state i as p_i. Let's also call $p = \frac{3}{4}$

$$\begin{aligned} p_0 &= 0, \\ p_1 &= pp_2, \\ p_2 &= pp_4, \\ p_3 &= (1-p)p_1 + p, \\ p_4 &= (1-p)p_3 + p, \\ p_5 &= 1. \end{aligned}$$

Solving this linear system we find $p_1 = \frac{135}{247}$, $p_2 = \frac{180}{247}$, $p_3 = \frac{219}{247}$, $p_4 = \frac{240}{247}$.

Problem 10. A frog is travelling from point $A(0,0)$ to point $B(5,4)$ but each step can only be 1 unit up or 1 unit to the right. Additionally, the frog refuses to move three steps in the same direction consecutively. Compute the number of ways the frog can move from A to B.

Answer 10. 45.

Wall Street Oasis

Here I am randomly selecting and solving problems from Wall Street Oasis website.

Problem 1. What is the degree difference on the hands of a clock at 3:15?

Answer 1. 7.5 degrees.

Solution 1. Obviously minute hand traveled exactly 90^o, so the difference

$$\Delta\theta = \left[\frac{3*60+15}{12*60} - \frac{1}{4}\right] * 360^o = \frac{5}{4} * 6^o = 7.5^o.$$

Problem 2. Suppose you have a $3 \times 3 \times 3$ cube. You paint the outside of the cube before cutting it into $1 \times 1 \times 1$ cubes. I pick a random $1 \times 1 \times 1$ cube from the pile of cubes and roll it. What is the probability that a painted side is face up?

Answer 2. 1/3.

Solution 2. There 4 types of $1 \times 1 \times 1$ cubes: 8 cubes that have three sides painted, 12 cubes that have two sides painted, 6 cubes that have single side painted, and 1 unpainted cube. Hence probability of ending up with painted face is

$$P = \frac{1}{27}\left[0 + 6 \cdot \frac{1}{6} + 12 \cdot \frac{1}{3} + 8 \cdot \frac{1}{2}\right] = \frac{1}{3}.$$

Turns out that if you repeat this process with $n \times n \times n$ cube, probability would be $\frac{1}{n}$.

Linearity of expectation

These problems are not from SIG test. They are compiled based on common solution technique.

Problem 1. A spinner has three regions, and the probabilities of landing in each region are $\frac{1}{4}$, $\frac{1}{4}$, and $\frac{1}{2}$. Compute the expected number of spins it would take to land in the same region twice in a row.

Answer 1. 15/4.

Solution 1. Let's label regions A, B, and C. Probabilities of landing on each regions are $p_A = p_B = \frac{1}{4}$ and $p_C = \frac{1}{2}$. Let x_A be the random variable defined as follows. We just landed on region A on the last spin and we a going to spin until we land on region A twice, the number of spins required to see AA is the value of x_A. If we landed on A again immediately, then $x_A = 1$. If we got $BACACBAA$, then $x_A = 8$, etc. Similarly we define x_B and x_C. Finally x is the random variable equal to the number of spins until we land on the same region twice (assuming that there were no prior spins). We are asked to find $X = \mathbb{E}[x]$.

$$\begin{aligned}
X &= p_A(1 + X_A) + p_B(1 + X_B) + p_C(1 + X_C),\\
&= 1 + p_A X_A + p_B X_B + p_C X_C,\\
X_A &= p_A + p_B(1 + X_B) + p_C(1 + X_C) = 1 + p_B X_B + p_C X_C,\\
X_B &= p_A(1 + X_A) + p_B + p_C(1 + X_C) = 1 + p_A X_A + p_C X_C,\\
X_C &= p_A(1 + X_A) + p_B(1 + X_B) + p_C = 1 + p_A X_A + p_B X_B
\end{aligned}$$

Solving linear system of equations we find $X_A = X_B = 3$ and $X_C = \frac{5}{2}$. So finally, $X = \frac{15}{4}$.

Problem 2. Suppose you are rolling a fair, 6-sided die repeatedly.

(a) What is the expected number of rolls until you get two 3's in a row (counting both 3's)?

(b) What is the expected number of rolls until you get a 3 followed by either a 3 or a 4?

(c) What is the expected number of rolls until you get a 3 followed by a 4?

Solution 2.

(a) Let x be the random variable we're interested in. x is the number rolls until we see two consecutive 3's (starting from no prior roll). Let x_3 be the number of rolls until we see two 3's one after another, assuming that the last roll was a 3. Let $p = 1/6$.

$$\begin{aligned}
\mathbb{E}[x] &= p(1 + \mathbb{E}[x_3]) + (1 - p)(1 + \mathbb{E}[x]),\\
\mathbb{E}[x_3] &= p + (1 - p)(1 + \mathbb{E}[x]).
\end{aligned}$$

Solving for $\mathbb{E}[x]$, I find $\mathbb{E}[x] = (1 + p)/p^2 = 42$.

(b) Similarly, let X be the sought random variable and x_3 as defined above.

$$\begin{aligned}
\mathbb{E}[x] &= p(1 + \mathbb{E}[x_3]) + (1 - p)(1 + \mathbb{E}[x]),\\
\mathbb{E}[x_3] &= 2p + (1 - 2p)(1 + \mathbb{E}[x]).
\end{aligned}$$

From here $\mathbb{E}[x] = (1 + p)/2p^2 = 21$.