PROBLEM SET #7
"Phonons"
due on 5/30/2021

Physics–172 / Applied Physics–272
Introduction to Solid State Physics
Spring quarter, 2022

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Problem 7.1 (Phonon chain — **5 points**). Consider one–dimensional chain of atoms (embedded in one–dimensional space) connected through springs with stiffness K. There are two alternating types of atoms with masses m_1 and m_2, distance between the atoms of the same mass is a (see Figure 1).

1. (**3 points**) Find dispersion relation $\omega(k)$ for the frequency of oscillations of atoms in one–dimensional chain
2. (**1 point**) Find sound velocity.
3. (**1 points**) What happens in the limit $m_1 = m_2$?

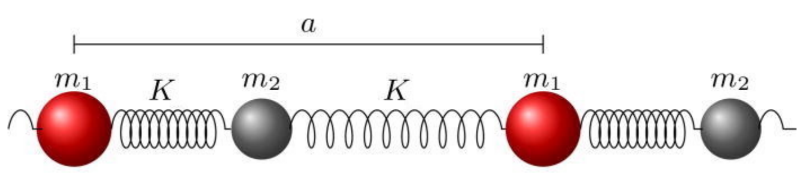

Figure 1: Model of 1D crystal.

Problem 7.2 (Phonon heat capacity — **4 points**). Phonons have linear energy–momentum relation $\varepsilon_{\mathbf{k}} \equiv \hbar\omega_{\mathbf{k}} = c|\mathbf{k}|$, where c is the speed of sound (we only consider acoustic phonons and disregard optical phonons). While real phonons have linear dispersion only at the center of the Brillouin zone, Debye model assumes that such relation holds for all momenta such that $|\mathbf{k}| < k_D$, where k_D is Debye wavevector, which is determined from the following normalization condition. One acoustic branch can host as many phonons as there are primitive cells (phonons are bosons — each one occupies its own slot in momenta space).

$$N_c = \int_{|\mathbf{k}|<k_D} \frac{V d^3\mathbf{k}}{(2\pi)^3}.$$

Here N_c is the number of crystal's primitive cells. Number of phonons, however, is not fixed (not conserved), hence chemical potential $\mu = 0$.

There are three acoustic modes in three–dimensional solid, as many as there are displacement directions. Hence total energy is

$$E = 3\int_{|\mathbf{k}|<k_D} \frac{V d^3\mathbf{k}}{(2\pi)^3}\, \varepsilon_{\mathbf{k}} n_B(\varepsilon_{\mathbf{k}}), \qquad n_B(\varepsilon) \equiv \left(\exp\frac{\varepsilon}{k_B T} - 1\right)^{-1}.$$

1. (**1 point**) Find expression for the Debye wavevector and corresponding energy scale — Debye temperature $\Theta = \hbar c k_D / k_B$.

2. (**1 point**) Write down an expression for the heat capacity in the following form

$$C_V = \left(\frac{\partial E}{\partial T}\right)_V = N_c k_B \times F\left(\frac{\Theta}{T}\right),$$

where F is some dimensionless function. Find asympotic expression for the C_V at high temperatures $T \gg \Theta$ (leading term only). Does your answer make sense from the point of view of equipartition theorem?

3. (**2 point**) Find asympotic expression for the C_V at low temperatures $T \ll \Theta$, i.e. show that

$$C_V \propto N k_B \left(\frac{T}{\Theta}\right)^3, \quad T \ll \Theta$$

and calculate the prefactor. Find characteristic values for Debye and Fermi temperatures of metals online and compare phonon heat capacity to cubic–in–T contribution from electrons.