X Assigment on «Applications in Theoretical Physics»

Problem X.1 (Wick's theorem)**.** Consider a function of order parameters φ_1, φ_2,..., all of which are normally distributed with zero mean and known variance $\langle\varphi^2\rangle$, so that

$$\langle\exp\varphi\rangle = \int d\varphi \exp\left(-\frac{\varphi^2}{2\langle\varphi^2\rangle}\right)\exp\varphi = \exp\left[\frac{1}{2}\langle\varphi^2\rangle\right].$$

Your task is to deal with expression containing exponential and trigonometric functions of order parameters, such as

$$\langle\cos\varphi_1\cos\varphi_2\rangle = \exp\left[-\langle\varphi^2\rangle\right].$$

Problem X.2 (Heisenberg antiferromagnet)**.** Consider Heisenberg antiferromagnetic Hamiltonian with periodic boundary conditions

$$\hat{H} = \sum_{k=1}^{N}\hat{\mathbf{s}}_k\cdot\hat{\mathbf{s}}_{k+1}, \quad N+1=1.$$

Find the spectrum of $\hat{H}$ for $s=\frac{1}{2}$ and $N = 2, 3, \ldots$, as well as for $s = 1$ and $N = 2, 3, \ldots$ How large N can you achieve?

Extrapolate the ground state energy divided by number of spins E_0/N and the value of the gap

$$\Delta \eqqcolon \min_{E\neq E_0} E - E_0$$

to large $N\to\infty$ for $s=\frac{1}{2}$ as well as $s=1$. Answer the following questions:

1) Is it correct that for $s=\frac{1}{2}$ $\lim\limits_{N\to\infty} E_0/N = \frac{1}{4}-\ln 2$?
2) What is the value of $\lim\limits_{N\to\infty} E_0/N$ for $s=1$?
3) Does the gap closes with $N\to\infty$ for $s=\frac{1}{2}$? for $s=1$?

Problem X.3 (Landau levels in graphene)**.** In the tight binding approximation effective Hamiltonian of graphene can be written as

$$H = \begin{pmatrix} 0 & \sum\limits_{n=1}^{3} e^{i\mathbf{k}\boldsymbol{\delta}_n} \\ \sum\limits_{n=1}^{3} e^{-i\mathbf{k}\boldsymbol{\delta}_n} & 0 \end{pmatrix} \qquad \boldsymbol{\delta}_1 = \frac{1}{2}\begin{pmatrix}\sqrt{3}\\-1\end{pmatrix} \quad \boldsymbol{\delta}_2 = \begin{pmatrix}0\\1\end{pmatrix} \quad \boldsymbol{\delta}_3 = \frac{-1}{2}\begin{pmatrix}\sqrt{3}\\1\end{pmatrix}$$

Presence of magnetic field can be accounted for via Peierls substitution $\mathbf{k}\mapsto\mathbf{k}-\frac{e}{c}\mathbf{A}$, where $\mathbf{A}$ is a vector potential. Find energy levels in graphene in magnetic field as a function of $\mathbf{k}$ for different orientations of the field $\mathbf{B}\parallel Oz$, Ox, Oy.

Problem X.4 (Hubbard model)**.** One of the rich model describing different phenomena in solid state physics is Hubbard model

$$\hat{H} = -t\sum_{\langle ij\rangle}\sum_{\sigma} c^+_\sigma(i)c_\sigma(j) + u\sum_i c^+_\uparrow(i)c_\uparrow(i)c^+_\downarrow(i)c_\downarrow(i).$$

Fermions live on $N\times N$ lattice with periodic boundary condition (try $N = 2, 3$). Here, I simply ask you to find quantum–statistically averaged number of fermions on the site

$$\left\langle c^+_\uparrow(i)c_\uparrow(i)\right\rangle \propto \operatorname{tr}\left[e^{-\beta\hat{H}}c^+_\uparrow(i)c_\uparrow(i)\right]$$

as a function of temperature $T=\frac{1}{\beta}$.

Problem X.5 (Over–barrier reflection)**.** Find the transmission coefficient for the over–barrier reflection in the potential $V(x) = -x^4$. In other words, you need to obtain solution of

$$\left[\frac{d^2}{dx^2} + x^4 + \varepsilon\right]\psi(x) = 0$$

that asymptotically behaves as

$$\psi(x) \sim t \cdot \frac{\exp \int_0^x \sqrt{\varepsilon - x^4}}{x}, \quad x \to \infty.$$

and then connect it with the behavior at the left end.

$$\psi(x) \sim \frac{\exp \int_0^x \sqrt{\varepsilon - x^4}}{x} + r \cdot \frac{\exp - \int_0^x \sqrt{\varepsilon - x^4}}{x}, \quad x \to -\infty.$$

Problem X.6 (Anderson model)**.** Simplest model describing impurities in conductors uses Hamiltonian

$$H = \sum_i \varepsilon_i c_i^+ c_i - t \sum_{\langle ij \rangle} c_i^+ c_j,$$

where fermion operators c_i are located on a line of size N with periodic boundary conditions. Disorder is presented via local «energies», which are uniformly distributed in $[-W, W]$ interval. Find (averaged) density of states in such model.