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АБСОЛЮТНЫЙ КОЭФФИЦИЕНТ ПУАССОНА В ДВУМЕРНЫХ КРИСТАЛЛИЧЕСКИХ МЕМБРАНАХ (ABSOLUTE POISSON RATIO OF 2D CRYSTALLINE MEMBRANES)

(магистерская диссертация)

Студент:
Сайкин Давид Рустамович

____
(подпись студента)

Научный руководитель:
Бурмистров Игорь Сергеевич,
д-р физ.-мат. наук

____
(подпись научного руководителя)

Консультант *(при наличии)*:
____
____
(подпись консультанта)

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MASTER'S THESIS

Absolute Poisson ratio of two–dimensional crystalline membranes

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Student ______________________

David Saykin

Supervisor ______________________

Igor Burmistrov

Professor

Co–Supervisor ______________________

Mikhail Skvortsov

Associate Professor

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David Saykin

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ABSTRACT

Fundamental theory of elasticity of two–dimensional materials is hard to develop since interaction between phonons that is essential for stability of two–dimensional crystal is strongly non–linear, thus it could not ever be fully comprehended analytically, and hard to study numerically. I develop perturbation theory in order to describe universal anomalous Hooke's law and auxetic Poisson ratio in two–dimensional crystalline membranes. I find that actual critical indices may dramatically differ from values given by self–consistent screening approximation.

Research Supervisor
Name: Igor Burmistrov
Degree: Doctor of Science
Title: Professor

Research Co–Supervisor
Name: Mikhail Skvortsov
Degree: Doctor of Science
Title: Associate Professor

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Contents

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Chapter 1

Introduction

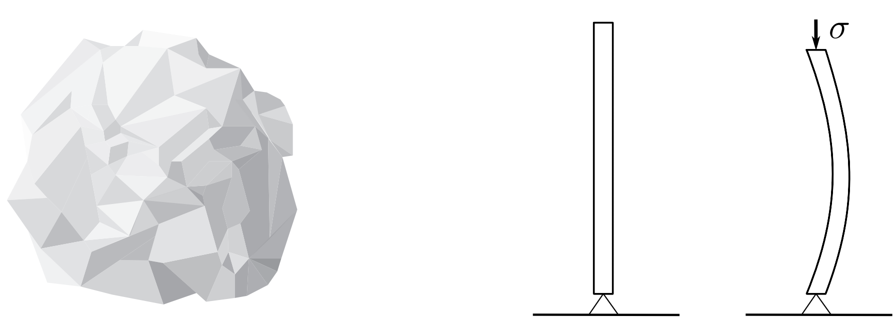
1.1 Motivation

Theory of elasticity of two–dimensional crystalline membranes as we know it today has been developing since 1980s [1] and is able to qualitatively describe some unique two–dimensional elastic phenomena such as crumpling and buckling [2], yet there are many qualitative and quantitative problems unsolved. Such theory mostly served to describe polymerized membranes in biological matter (e.g. red blood cells), however recently interest to that field of study has been revived with experimental observation of graphene — single atom layer of carbon. That is exactly an application I would keep in mind below.

Most prominent and curious phenomena of elastic physics of graphene are anomalous Hooke's law and negative Poisson ratio. On Figure 2a you can see experimentally measured stress σ versus strain ξ dependence of free–hanging graphene sheet [3]. That dependence is believed to be described by power law

$$\delta\xi = \frac{L_\sigma - L_{\sigma=0}}{L_{\sigma=0}} \propto \frac{\sigma}{Y_0}\left(\frac{\sigma}{\sigma_*}\right)^{\alpha-1}, \quad \sigma \ll \sigma_*. \tag{1.1}$$

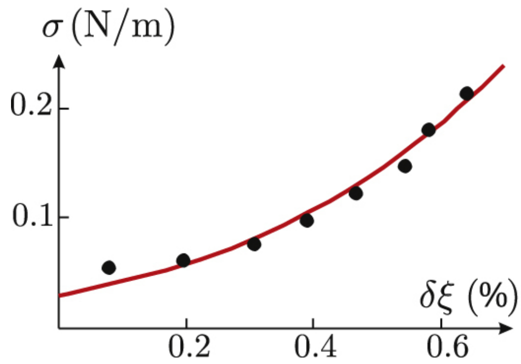
Here $Y_0 \simeq 22$ eV$\cdot$Å^{-2} is (bare) graphene Young modulus and $\sigma_* \simeq .5$ N$\cdot$m^{-1} is characteristic scale of non–linear regime. Exponent $\alpha > 0$ is not known exactly, but several analytical approximations and numerical studies suggest that its value lies in range $.6 \lesssim \alpha_{\text{clean}} \lesssim .7$ for the clean membranes and $\alpha_{\text{dis}} \lesssim .3$ for disordered membranes (e.g. graphene with dislocations).



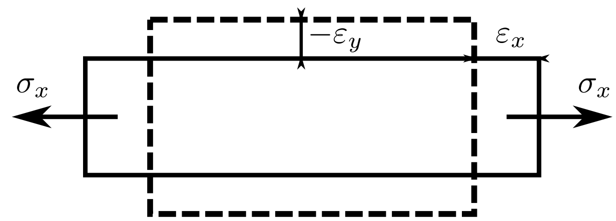
(a) Crumpled piece of paper.

(b) Buckling transition.

Figure 1: Illustration of crumpling and buckling phenomena.



(a) Experimental evidence of anomalous Hooke's law $\alpha_{\text{fit}} \approx .1$ in graphene [8].



(b) Definition of Poisson ratio.

Figure 2: Unusual elastic phenomena peculiar to two–dimensional membranes.

Poisson ratio (PR) is defined as the strain ratio in perpendicular directions when stress is applied only along one direction (see Figure 2b). In conventional linear regime it is given by material dependent lame coefficients and is usually positive, which means that stretching in one direction leads to shrinking in another.

$$\nu_0 = -\frac{\varepsilon_y}{\varepsilon_x} = \frac{\lambda}{2\mu + (D-1)\lambda}.$$

For graphene $\nu_0 \approx .1$. However, it so happens that in non–linear regime (1.1) Poisson ration becomes universal for all two–dimensional crystalline surfaces number, which is believed to be negative [4]. Also in such regime in addition to absolute Poisson ratio a differential one is defined.

$$\nu = -\frac{\xi_y(\sigma_x, 0) - \xi_y(0,0)}{\xi_x(\sigma_x, 0) - \xi_x(0,0)}, \qquad \nu^{\text{diff}} = -\left.\frac{\partial \xi_y(\sigma_x, \sigma)/\partial \sigma_x}{\partial \xi_y(\sigma_x, \sigma)/\partial \sigma_x}\right|_{\sigma_x = \sigma}.$$

Universal value of Poisson ratio in the regime $\sigma_L \ll \sigma \ll \sigma_*$ has been believed [5, 6] to be unprecedentedly close to $\nu_{\text{SCSA}} = -\frac{1}{3}$, however it was recently explicitly shown [7] that there is no reason to believe in exact relation $\nu^{\text{diff}} = \nu_{\text{SCSA}}$.

Both anomalous Hooke's law and auxetic phenomena may be understood with the simple notion of ripples – small waves on the membrane surface, which are always present due to thermal fluctuations (see Figure 3). At finite temperature free-standing surface is nearly flat but due to folds its projected area is smaller that of the same membrane at absolute zero $T = 0$, i. e. it is shrunk in both directions. Thus, application of external tensions will at first flatten the surface and only after that stretch the crystalline structure, which means that it is easier to spring out the membrane at small stresses. Also such initial thermal decrease of projected area is naturally isotropic, so flattening by pulling in any direction leads to expansion in both, i.e. negative Poisson ratio.

1.2 Problem statement

In my thesis I am going to study elastic properties of two–dimensional crystalline membrane in non-linear (universal) stress regime $\sigma \ll \sigma_*$. In order to do that at first I am going to derive model of 2D membrane under the finite stress (Chapter 2) and develop a controlled perturbation theory (Chapter 3) in order to obtain analytical expressions depicting anomalous Hooke's law. Specifically, I am going to use perturbative approach to answer the following questions.

1. How does the critical exponent α from anomalous Hooke's law differ from the known approximate values, namely, self–consistent screening approximation[9]? (Section 3.3)

2. Can the value of Poisson ratio ν in universal regime be expressed in terms of α or is it an independent critical index? How does it differ from its value given by self–consistent screening approximation [10]? (Section 3.4)

I am going to reinforce my conclusions based on perturbation theory with numerical Monte Carlo simulations using the effective energy functional I have derived. In Chapter 4 I explain how simulations are made and review and discuss existing articles reporting on numerical values of critical indices.

Appendix A contains calculations that are too lengthy to be present in main text.

Conclusions and results summary are present in Chapter 5.

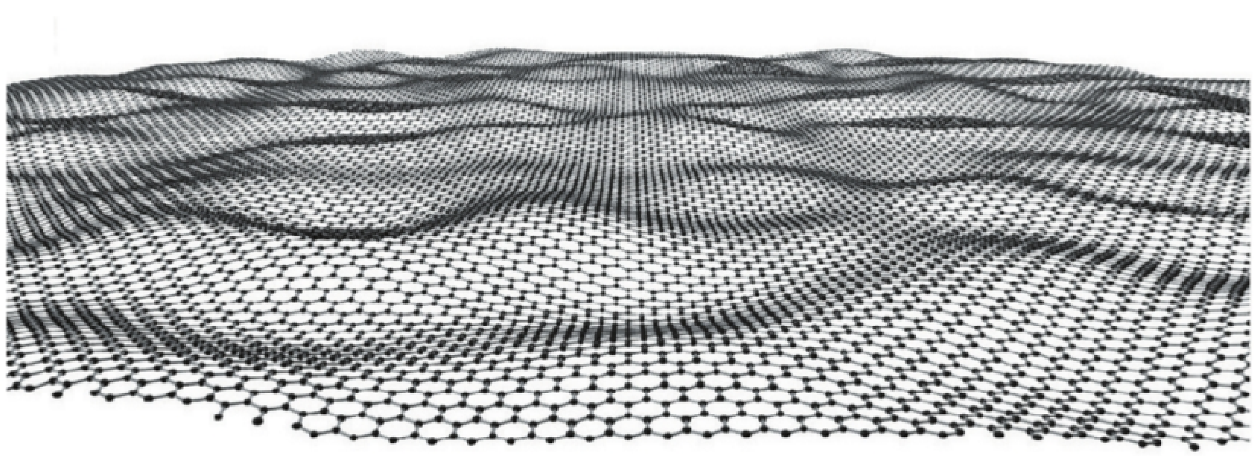


Figure 3: Graphene lattice with ripples [source].

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Chapter 2
Theoretical model

2.1 Free Energy of two–dimensional membrane

The model used to describe two–dimensional crystals (tethered surfaces) for the past several decades [11] takes into account surface bending energy and bonding potential [12, Ch. 6].

$$E = \bar{\varkappa} \sum_{\langle \mathbf{xy} \rangle} (1 - \mathbf{n_x} \cdot \mathbf{n_y}) + \sum_{\langle \mathbf{xy} \rangle} V\left(|\mathbf{r_x} - \mathbf{r_y}| \right).$$

Here $\mathbf{x}$ is a discrete index running through the $D = 2$–dimensional lattice nodes and sum is taken of neighboring atoms, normal vectors $\mathbf{n_x}$ are located at the center of adjacent cell (see Figure 4). At zero temperature $T = 0$ membrane is presumed to be perfectly flat, at finite temperature its profile is given by $\mathbf{r}(\mathbf{x})$ function.

$$\mathbf{r}(\mathbf{x}) = \begin{pmatrix} x_1 + u_1(\mathbf{x}) \\ x_2 + u_2(\mathbf{x}) \\ h(\mathbf{x}) \end{pmatrix} \qquad \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2.$$

In the limit of continuous media energy of smooth (without self–intersections) nearly flat surface could be expanded in gradients of $\mathbf{r}(\mathbf{x})$.

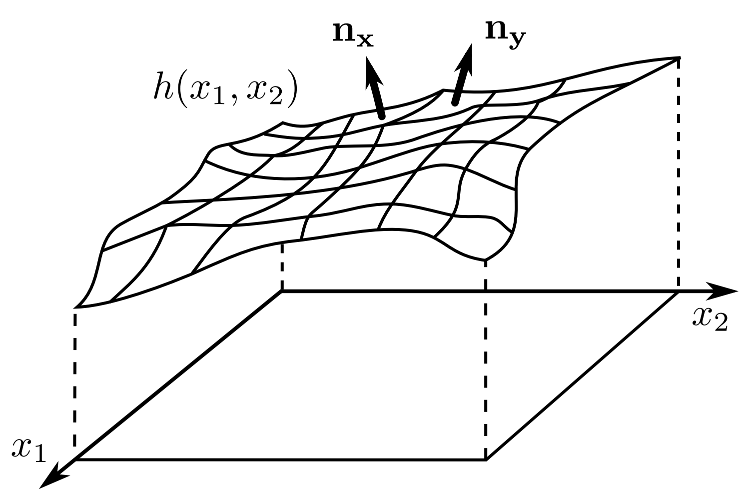


Figure 4: Parametrization of membrane in plane phase.

Bending energy Membrane surface is parametrized with equation $z = h(\mathbf{x}+\mathbf{u}(\mathbf{x}))$. As it will be seen below it is enough to approximate $z \approx h(\mathbf{x})$, then normal vector expressed as follows.

$$\mathbf{n} = \frac{1}{\sqrt{1+(\nabla h)^2}} \begin{pmatrix} -\partial_1 h \\ -\partial_2 h \\ 1 \end{pmatrix} \qquad h_{\mathbf{x}+\mathbf{e}_i} \approx h_{\mathbf{x}} + (\mathbf{e}_i, \nabla) h_{\mathbf{x}} + \frac{1}{2}(\mathbf{e}_i, \nabla)^2 h_{\mathbf{x}}.$$

In the equation below $\tilde{\varkappa}$ is proportional to $\bar{\varkappa}$. Proportionality coefficient depends on the lattice type and is equal to 1 for square lattice and $1/\sqrt{3}$ for graphene hexagonal lattice.

$$\mathbf{n}_{\mathbf{x}} \cdot \mathbf{n}_{\mathbf{x}+\mathbf{e}_i} = \sqrt{\frac{1+(\nabla h_{\mathbf{x}})^2}{1+(\nabla h_{\mathbf{y}})^2}} \left[1 + \frac{(\nabla h_{\mathbf{x}}, \nabla (h_{\mathbf{y}} - h_{\mathbf{x}}))}{1+(\nabla h_{\mathbf{x}})^2}\right]_{\mathbf{y}=\mathbf{x}+\mathbf{e}_i} \approx 1 - \frac{1}{2}(\nabla_\alpha (\mathbf{e}_i \nabla) h_{\mathbf{x}})^2.$$

$$\bar{\varkappa} \sum_{\langle \mathbf{x}\mathbf{y} \rangle} (1 - \mathbf{n}_{\mathbf{x}} \cdot \mathbf{n}_{\mathbf{y}}) \approx \frac{\tilde{\varkappa}}{2} \int d^D\mathbf{x} \, (\partial_\alpha \partial_\beta h_{\mathbf{x}})^2 = \int d^D\mathbf{x} \, \left[(\nabla^2 h_{\mathbf{x}})^2 - 2 \det(\partial_\alpha \partial_\beta h_{\mathbf{x}})\right].$$

The last two terms of are mean and Gaussian curvature, second term may be omitted since it is a perfect derivative and boundary contributions are negligible in thermodynamic limit.

Potential energy The same way potential energy can be expanded in displacement gradients.

$$V\left(|\mathbf{r}_{\mathbf{x}} - \mathbf{r}_{\mathbf{y}}|\right) = V(0) + \partial_\mu V(0) \left[(\mathbf{e}_i, \nabla) r^\mu_{\mathbf{x}} + \frac{1}{2}(\mathbf{e}_i, \nabla)^2 r^\mu_{\mathbf{x}} + \dots\right] + \dots$$

After averaging over lattice basis vectors I obtain expression (2.1) with coefficients t, $\tilde{\lambda}$, $\tilde{\mu}$ having some tensor structure. It happens because generally couplings like $(\partial_\alpha r_\alpha)^2$ and $(\partial_\alpha r_\beta)(\partial_\beta r_\alpha)$ and other quartic terms are also possible, however, in the text below full rotational invariance is assumed for simplicity.

Continuous limit Altogether, energy is written as

$$\mathcal{F}[\mathbf{r}] = \frac{1}{2} \int d^D\mathbf{x} \left[\tilde{\varkappa} \left(\nabla^2 h\right)^2 + t \left(\partial_\alpha r_\beta\right)^2 + \frac{\tilde{\mu}}{2} \left(\partial_\alpha \mathbf{r} \cdot \partial_\beta \mathbf{r}\right)^2 + \frac{\tilde{\lambda}}{4} \left(\partial_\alpha r_\beta\right)^4\right]. \qquad (2.1)$$

Sign of t governs transition between crumpled and plane phases as may be seen from mean–field consideration [1, 13]. For my purposes membrane could be considered deep in the plane phase ($t < 0$), then it is convenient to rescale $\mathbf{r} \mapsto \xi_0 \mathbf{r}$ and introduce $\varkappa = \xi_0^2 \tilde{\varkappa}$, $\mu = \xi_0^4 \tilde{\mu}$, $\lambda = \xi_0^4 \tilde{\lambda}$ with $\xi_0^2 = -t/(\tilde{\mu} + D\tilde{\lambda}/2)$.

$$\mathcal{F}[\mathbf{r}] = \frac{1}{2} \int d^D\mathbf{x} \left[\varkappa (\nabla^2 h)^2 + \frac{\mu}{2} (\partial_\alpha \mathbf{r} \cdot \partial_\beta \mathbf{r} - \delta_{\alpha\beta})^2 + \frac{\lambda}{4} (\partial_\alpha \mathbf{r} \cdot \partial_\alpha \mathbf{r} - D)^2\right]. \qquad (2.2)$$

2.1.1 External stress

In the presence of external stress free energy should obtain $-\sigma_{\alpha\beta}\delta\partial_\alpha r_\beta$ addition [1] similarly to term $-p\delta V$ in thermodynamics of gases.

$$\mathcal{F}_{\rm tot}[\mathbf{r},\sigma_{\alpha\beta}] = \mathcal{F}[\mathbf{r}] + \mathcal{F}_{\rm ext}[\mathbf{r},\sigma_{\alpha\beta}], \qquad \mathcal{F}_{\rm ext} = -\int d^D\mathbf{x}\, \sigma_{\alpha\beta}(\mathbf{x})\partial_\alpha r_\beta(\mathbf{x}).$$

Generally, one has to distinguish between coupling stress directly to strain tensor as it is done in [14, §3] and using more precise definition of work. As I will show below difference between these two options is negligible.

In this work I will only consider the case of the uniform stress $\sigma_{\alpha\beta} = \mathrm{const}(\mathbf{x})$. It is realized, for example, in the case of suspended rectangular or circular membrane to which uniform tension along the edge is applied. In each of these cases I can rewrite additional term as boundary integral.

$$\mathcal{F}_{\rm ext} = -\sigma_{\alpha\beta}\int_\Omega d^D\mathbf{x}\,\partial_\alpha r_\beta(\mathbf{x}) = -\sigma_{\alpha\beta}\oint_{\mathbf{x}\in\partial\Gamma} n_\alpha(\mathbf{x}) r_\beta(\mathbf{x}) dl.$$

I define stretching coefficients as the ratio of given length and initial length (at $T=0$) and shift integration variable in $\mathcal{F}_{\rm tot}$ in such way $r_\mu = \xi_{\mu\alpha}x_\alpha + \delta r_\mu$ that external term becomes $\mathbf{r}$–independent, thus, may be excluded from action.

$$\mathcal{F}_{\rm tot}[\xi_{\mu\alpha}x_\alpha + \delta r_\mu, \sigma_{\alpha\beta}] = \mathcal{F}[\xi_{\mu\alpha}x_\alpha + \delta r_\mu] - L^D\xi_{\beta\alpha}\sigma_{\alpha\beta}$$

since $\delta r_\mu = 0$ at the boundary by construction. It is easier to see how that happens on the explicit examples below.

Rectangle Simplest possible case, which considered in the rest of the paper, is the sample of rectangular shape.

$$\Omega_{\rm rect} = \left[-\frac{L_x^0}{2}, \frac{L_x^0}{2}\right] \times \left[-\frac{L_x^0}{2}, \frac{L_x^0}{2}\right].$$

Strain tensor and stress term has the simplest possible form

$$\begin{aligned} \xi_{xx} &= \frac{L_x}{L_x^0}, & \xi_{yx} &= 0, \\ \xi_{xy} &= 0 & \xi_{yy} &= \frac{L_y}{L_y^0}. \end{aligned} \qquad \mathcal{F}_{\rm ext} = -\sigma_{xx}L_xL_y^0 - \sigma_{yy}L_yL_x^0.$$

Here L_x, L_y are the actual size of the membrane, i.e. are the sizes of the projected area, when L_x^0, L_y^0 are the sizes of the $\mathbf{x}$–grid.

Circle Another way to have uniform stress tensor is to impose circular geometry.

$$\Omega_{\text{circ}} = \{\mathbf{x} : |\mathbf{x}| < R_0\}.$$

Similarly, actual R is a function of stress.

$$\begin{aligned} \xi_{rr} &= \frac{R}{R_0}, & \xi_{\varphi r} &= 0, \\ \xi_{r\varphi} &= 0 & \xi_{\varphi\varphi} &= 0. \end{aligned} \qquad \mathcal{F}_{\text{ext}} = -\sigma_{rr}\pi R R^0.$$

General approach In general case of non–uniform stress, similar expressions may be obtained in thermodynamic limit. Following the idea of the paper [7], I am going to work in the setting with fixed spatial derivative $\langle \partial_\alpha r_\beta \rangle = m_{\alpha\beta}(\mathbf{x})$ rather than given stress tensor $\sigma_{\alpha\beta}(\mathbf{x})$. I define Legendre transform of the functional $\tilde{\mathcal{F}}_{tot}$ according to the usual rule

$$\tilde{\mathcal{F}}_{\text{tot}}[m_{\alpha\beta}] = \left[\mathcal{F}_{tot}[\sigma_{\alpha\beta}] + \int d^D\mathbf{x}\, \sigma_{\alpha\beta}(\mathbf{x}) m_{\alpha\beta}(\mathbf{x})\right]_{\sigma_{\alpha\beta}=\sigma_{\alpha\beta}(m_{\alpha\beta})} \tag{2.3}$$

where $\sigma_{\alpha\beta}$ is expressed in terms of $m_{\alpha\beta}$ through the inverse of the relation

$$m_{\alpha\beta} = \langle \partial_\alpha r_\beta \rangle_{\mathcal{F}_{\text{tot}}} = -\frac{1}{L^D}\frac{\delta \mathcal{F}_{\text{tot}}}{\delta \sigma_{\alpha\beta}}.$$

In its turn, such functional has to satisfy

$$\sigma_{\alpha\beta} = \frac{1}{L^D}\frac{\delta \tilde{\mathcal{F}}_{\text{tot}}}{\delta m_{\alpha\beta}}.$$

In the article [7] it is stated that $\tilde{\mathcal{F}}_{\text{tot}}[m_{\alpha\beta}]$ coincides with $\mathcal{F}$ evaluated with the constriction $\langle \partial_\alpha r_\beta \rangle = m_{\alpha\beta}$, such action can written as

$$e^{-\mathcal{F}_1/T} = e^{-\mathcal{F}/T}\delta[\partial_\alpha r_\beta - m_{\alpha\beta}].$$

Or equivalently with integral over $s_{\alpha\beta}$ runs along imaginary line.

$$\begin{aligned} \mathcal{F}_1[\mathbf{r}, m_{\alpha\beta}] &= -T\ln\int D[s_{\alpha\beta}]\exp\left[-\frac{\mathcal{F}}{T} + \int \frac{s_{\alpha\beta}}{T}\left(\partial_\alpha r_\beta - m_{\alpha\beta}\right) d^D\mathbf{x}\right] \\ &= -T\ln\int D[s_{\alpha\beta}]\exp\left[-\frac{\mathcal{F}_{\text{tot}}[s_{\alpha\beta}]}{T} - \int \frac{s_{\alpha\beta}}{T} m_{\alpha\beta} d^D\mathbf{x}\right] \end{aligned}$$

Evaluating integral over $s_{\alpha\beta}$ in saddle approximation I come to (2.3).

2.1.2 Harmonic approximation

It was shown in previous section that in order to describe free–standing membrane with fixed uniform applied stress $\sigma_{\alpha\beta}$, one can consider action

$$\mathcal{F}[\mathbf{r}] = \frac{1}{2}\int d^D\mathbf{x}\left[\varkappa(\nabla^2 h)^2 + \frac{\mu}{2}(\partial_\alpha\mathbf{r}\cdot\partial_\beta\mathbf{r} - \delta_{\alpha\beta})^2 + \frac{\lambda}{4}(\partial_\alpha\mathbf{r}\cdot\partial_\alpha\mathbf{r} - D)^2\right]. \tag{2.4}$$

with shifted variable $\mathcal{F} = \mathcal{F}[\xi_{\mu\alpha}x_\alpha + \delta r_\mu]$ and imposed conditions

$$\delta r_{x,y}|_{\partial\Omega} = 0, \qquad \frac{\delta\mathcal{F}}{\delta\xi_{\beta\alpha}(\mathbf{x})} = \sigma_{\alpha\beta}(\mathbf{x}). \tag{2.5}$$

In order to simplify action I make substitution

$$\mathbf{r}(\mathbf{x}) = \begin{pmatrix}\xi_1 x_1 + u_1(\mathbf{x})\\ \xi_2 x_2 + u_2(\mathbf{x})\\ h(\mathbf{x})\end{pmatrix}$$

and expand all therms in (2.4). Let me define a strain tensor as

$$u_{\alpha\beta} = \frac{1}{2}\left(\xi_{(\alpha)}\partial_\beta u_\alpha + \xi_{(\beta)}\partial_\alpha u_\beta + \partial_\alpha\mathbf{u}\cdot\partial_\beta\mathbf{u} + \partial_\alpha h\cdot\partial_\beta h\right)$$

then I can rewrite the following terms

$$\begin{aligned}\partial_\alpha\mathbf{r}\cdot\partial_\beta\mathbf{r} &= (\xi_{(\gamma)}\delta_{\alpha\gamma} + \partial_\alpha u_\gamma + \partial_\alpha h_\gamma)(\xi_{(\gamma)}\delta_{\beta\gamma} + \partial_\beta u_\gamma + \partial_\beta h_\gamma)\\ &= (\xi_{(\alpha)}\delta_{\alpha\gamma} + \partial_\alpha u_\gamma)(\xi_{(\beta)}\delta_{\beta\gamma} + \partial_\beta u_\gamma) + \partial_\alpha h\cdot\partial_\beta h\\ &= \xi_{(\alpha)}\xi_{(\beta)}\delta_{\alpha\beta} + 2u_{\alpha\beta}.\\ \partial_\alpha\mathbf{r}\cdot\partial_\alpha\mathbf{r} &= \xi_\alpha^2 + 2u_{\alpha\alpha},\end{aligned}$$

where Einstein summation rule is presumed as always, so that, for example, $\xi_\alpha^2 = \xi_\alpha\xi^\alpha = \xi_x^2 + \xi_y^2$.

Here I define proportional deformations ε_α in the following way supposing that $\xi_\alpha \approx 1$.

$$\varepsilon_\alpha \rightleftharpoons \frac{\xi_\alpha\xi_{(\alpha)} - 1}{2} \approx \xi_\alpha - 1 = \frac{\Delta_\alpha L}{L}, \quad \boldsymbol{\varepsilon} = \frac{1}{2}\begin{pmatrix}\xi_x^2 - 1\\ \xi_y^2 - 1\end{pmatrix}$$

Local proportional deformations for uniaxial deformations are given by diagonal elements of strain tensor, I denote them as $\frac{1}{2}K_\alpha$.

$$K_\alpha \rightleftharpoons 2u_{\alpha(\alpha)}, \quad \mathbf{K} = \begin{pmatrix}2u_{xx}\\ 2u_{yy}\end{pmatrix}$$

I modify expressions further.

$$
\begin{aligned}
\frac{1}{4}(\partial_\alpha \mathbf{r}\cdot\partial_\beta \mathbf{r}-\delta_{\alpha\beta})^2 &= \frac{1}{4}(\xi_{(\alpha)}\xi_{(\beta)}\delta_{\alpha\beta}-\delta_{\alpha\beta}+2u_{\alpha\beta})(\xi_{(\alpha)}\xi_{(\beta)}\delta_{\alpha\beta}-\delta_{\alpha\beta}+2u_{\alpha\beta}) \\
&= (\varepsilon_{(\alpha)}\delta_{\alpha\beta}+u_{\alpha\beta})(\varepsilon_{(\alpha)}\delta_{\alpha\beta}+u_{\alpha\beta}) \\
&= \varepsilon_\alpha^2+2\varepsilon_{(\alpha)}u_{\alpha\alpha}+u_{\alpha\beta}^2 \\
&= \varepsilon_\alpha(\varepsilon_\alpha+K_\alpha)+u_{\alpha\beta}^2. \\
\frac{1}{4}(\partial_\alpha \mathbf{r}\cdot\partial_\alpha \mathbf{r}-D)^2 &= \frac{1}{4}(\xi_\alpha^2-\delta_{\alpha\alpha}+2u_{\alpha\alpha})^2 \\
&= \left(\varepsilon_\alpha+\frac{1}{2}K_\alpha\right)1_{\alpha\beta}\left(\varepsilon_\beta+\frac{1}{2}K_\beta\right) \\
&= \varepsilon_\alpha 1_{\alpha\beta}(\varepsilon_\beta+K_\beta)+(u_{\alpha\alpha})^2.
\end{aligned}
$$

Here $1_{\alpha\beta}$ is 2×2 matrix with all elements equal to unity. Sum of these terms is

$$
\frac{\mu}{4}(\partial_\alpha \mathbf{r}\cdot\partial_\beta \mathbf{r}-\delta_{\alpha\beta})^2+\frac{\lambda}{8}(\partial_\alpha \mathbf{r}\cdot\partial_\alpha \mathbf{r}-D)^2=\varepsilon_\alpha\left[\mu\delta_{\alpha\beta}+\frac{\lambda}{2}1_{\alpha\beta}\right](\varepsilon_\beta+K_\beta)+\mu(u_{\alpha\alpha})^2+\frac{\lambda}{2}u_{\alpha\beta}^2.
$$

So that free energy is given by (compare with [14, (4.1)])

$$
\mathcal{F}[\mathbf{u},h,\varepsilon]=\int d^D\mathbf{x}\left\{\frac{\varkappa}{2}(\nabla^2 h)^2+\mu(u_{\alpha\alpha})^2+\frac{\lambda}{2}u_{\alpha\beta}^2+\frac{1}{2}\varepsilon_\alpha M_{\alpha\beta}(\varepsilon_\beta+\overline{K_\beta})\right\}, \tag{2.6}
$$

where I have introduced matrix

$$
M_{\alpha\beta}\rightleftharpoons 2\mu\delta_{\alpha\beta}+\lambda 1_{\alpha\beta},\qquad M=\begin{pmatrix}2\mu+\lambda & \lambda\\ \lambda & 2\mu+\lambda\end{pmatrix}. \tag{2.7}
$$

and switched K_α with their space averaged values.

$$
\overline{K}_\beta=\int\frac{d^D\mathbf{x}}{L^D}K_\beta=\int d^D\mathbf{x}\left(\partial_\beta\mathbf{u}\cdot\partial_{(\beta)}\mathbf{u}+\partial_\beta\mathbf{h}\cdot\partial_{(\beta)}\mathbf{h}\right)
$$

since $\mathbf{u}(0)=\mathbf{u}(L)=0$.

Harmonic approximation In order to move on, I first consider harmonic limit. I define Green functions as

$$
\begin{aligned}
L^{-D}\left\langle u^\alpha_{\mathbf{q}}u^\beta_{-\mathbf{q}}\right\rangle &= F^{(l)}_{\mathbf{q}}\frac{q_\alpha q_\beta}{q^2}+F^{(t)}_{\mathbf{q}}\left(\delta_{\alpha\beta}-\frac{q_\alpha q_\beta}{q^2}\right), \\
L^{-D}\left\langle h_{\mathbf{q}}h_{-\mathbf{q}}\right\rangle &= G_{\mathbf{q}}.
\end{aligned}
$$

For simplicity I set $\xi_x = \xi_y = \xi$, then in harmonic approximation [15, D2].

$$
\begin{aligned}
F^{(l)}_{\mathbf{q}} &= \frac{1}{(2\mu+\lambda)\xi^2 + (\mu+\lambda)(\xi^2-1)}\frac{1}{q^2} \\
F^{(l)}_{\mathbf{q}} &= \frac{1}{\mu\xi^2 + (\mu+\lambda)(\xi^2-1)}\frac{1}{q^2} \\
G_{\mathbf{q}} &= \frac{T}{\varkappa q^4 + (\mu+\lambda)(\xi^2-1)q^2}
\end{aligned}
$$

From here I see that for $q < q_\varkappa = \sqrt{\mu/\varkappa} \simeq 3\ \mathring{\mathrm{A}}^{-1}$ I can neglect[1] $uu \ll hh$ in expressions for strain tensor $u_{\alpha\beta}$ and K_α. Since $q_\varkappa$ is numerically of order ultraviolet cutoff $\frac{\pi}{a} \simeq 3\ \mathring{\mathrm{A}}^{-1}$ that is what is usually done [15].

Now it is suitable to rescale $\xi_{(\alpha)} u_\alpha \mapsto u_\alpha$, so that

$$
u_{\alpha\beta} = \frac{1}{2}\left(\partial_\beta u_\alpha + \partial_\alpha u_\beta + \partial_\alpha h \cdot \partial_\beta h\right) \qquad \overline{K}_\beta = \int \frac{d^D\mathbf{x}}{L^D}\left(\partial_{(\beta)} h \cdot \partial_\beta h\right).
$$

High d_c expansion Here I note that unharmonic terms contain no smallness which makes the problem of calculating functional integral intractable analytically. I introduce fictitious large parameter $d_c \gg 1$ — size of the $\mathbf{h}$ vector. Physical case correspond to $d_c = 1$, however, keeping arbitrary d_c helps to develop controllable perturbation theory which will be done in Chapter 3.

$$
\mathcal{F}[\mathbf{u},\mathbf{h}] = \frac{1}{2}\int d^D\mathbf{x}\left\{\varkappa(\nabla^2\mathbf{h})^2 + 2\mu(u_{\alpha\alpha})^2 + \lambda u_{\alpha\beta}^2 + M_{\alpha\beta}\left[\left(\varepsilon_\alpha + \frac{\overline{K}_\alpha}{2}\right)\left(\varepsilon_\beta + \frac{\overline{K}_\beta}{2}\right) - \frac{\overline{K}_\alpha\overline{K}_\beta}{4}\right]\right\}.
$$

Where strain tensor now written is

$$
u_{\alpha\beta} = \frac{1}{2}\left(\partial_\beta u_\alpha + \partial_\alpha u_\beta + \partial_\alpha \mathbf{h} \cdot \partial_\beta \mathbf{h}\right) \qquad \overline{K}_\beta = \int \frac{d^D\mathbf{x}}{L^D}\left(\partial_{(\beta)} \mathbf{h} \cdot \partial_\beta \mathbf{h}\right).
$$

[1] Conventional theory of elasticity neglects both u^2 and h^2, thus, making the action Gaussian.

2.1.3 Effective action for flexural phonons

In order to proceed I want to integrate out $\mathbf{u}$—modes.

$$e^{-\mathcal{F}_{\mathrm{eff}}[\mathbf{h}]/T} = \int D[\mathbf{u}] e^{-\mathcal{F}[\mathbf{u},\mathbf{h}]/T}.$$

I expand $u_{\alpha\beta}$ terms

$$\begin{aligned}
(u_{\alpha\beta})^2 &= \frac{1}{4}\left(\partial_\beta u_\alpha + \partial_\alpha u_\beta + \partial_\alpha \mathbf{h}\cdot\partial_\beta\mathbf{h}\right)^2 \\
&= \frac{1}{2}(\partial_\alpha u_\beta)^2 + \frac{1}{2}\partial_\alpha u_\beta\cdot\partial_\beta u_\alpha + \partial_\alpha u_\beta\cdot(\partial_\alpha\mathbf{h}\cdot\partial_\beta\mathbf{h}) + \frac{1}{4}(\partial_\alpha\mathbf{h}\cdot\partial_\beta\mathbf{h})^2 \\
(u_{\alpha\alpha})^2 &= \frac{1}{4}\left(2\partial_\alpha u_\alpha + \partial_\alpha\mathbf{h}\cdot\partial_\alpha\mathbf{h}\right)^2 \\
&= (\partial_\alpha u_\alpha)^2 + \partial_\beta u_\beta\cdot(\partial_\alpha\mathbf{h}\cdot\partial_\alpha\mathbf{h}) + \frac{1}{4}(\partial_\alpha\mathbf{h}\cdot\partial_\alpha\mathbf{h})^2
\end{aligned}$$

and divide energy into three parts

$$\mathcal{F} = \mathcal{F}_u + \mathcal{F}_{uh} + \mathcal{F}_h,$$

where $\mathcal{F}_u$ is quadratic in $\mathbf{u}$, $\mathcal{F}_{uh}$ is linear in $\mathbf{u}$ and $\mathcal{F}_h$ contains everything else.

$$\begin{aligned}
\mathcal{F}_u &= \frac{1}{2}\int d^D\mathbf{x}\left\{\mu\left[(\partial_\alpha u_\beta)^2 + \partial_\alpha u_\beta\cdot\partial_\beta u_\alpha\right] + \lambda(\partial_\alpha u_\alpha)^2\right\}. \\
\mathcal{F}_{uh} &= \int d^D\mathbf{x}\left\{\mu\partial_\alpha u_\beta\cdot(\partial_\alpha\mathbf{h}\cdot\partial_\beta\mathbf{h}) + \frac{\lambda}{2}\partial_\beta u_\beta\cdot(\partial_\alpha\mathbf{h}\cdot\partial_\alpha\mathbf{h})\right\}. \\
\mathcal{F}_h &= \frac{1}{2}\int d^D\mathbf{x}\left\{\varkappa(\nabla^2\mathbf{h})^2 + \frac{\mu}{2}(\partial_\alpha\mathbf{h}\cdot\partial_\beta\mathbf{h})^2 + \frac{\lambda}{4}(\partial_\alpha\mathbf{h}\cdot\partial_\alpha\mathbf{h})^2 + M_{\alpha\beta}\left[\ldots\right]\right\}.
\end{aligned}$$

I impose periodic boundary conditions since it is much easier to work with them than with zero boundary conditions. Producing Fourier transformation

$$\mathbf{u}(\mathbf{x}) = \int (d\mathbf{q}) e^{i\mathbf{q}\mathbf{x}}\mathbf{u}_\mathbf{q} \equiv \int \frac{d^D\mathbf{q}}{(2\pi)^D} e^{i\mathbf{q}\mathbf{x}}\mathbf{u}_\mathbf{q} = \frac{1}{L^D}\sum_\mathbf{q} e^{i\mathbf{q}\mathbf{x}}\mathbf{u}_\mathbf{q} \tag{2.8}$$

and exploiting that $\mathbf{u}$ fields are real $\mathbf{u}_{-\mathbf{q}} = \mathbf{u}^*_\mathbf{q}$, I come to expressions

$$\begin{aligned}
\mathcal{F}_u &= \frac{1}{2}\int (d\mathbf{q})(u^\alpha_\mathbf{q})^*\left\{\mu q^2\delta_{\alpha\beta} + (\mu+\lambda)q_\alpha q_\beta\right\}u^\beta_\mathbf{q}. \\
\mathcal{F}_{uh} &= \int (d\mathbf{q})\int d^D\mathbf{x}\, e^{i\mathbf{q}\mathbf{x}}(\partial_\alpha\mathbf{h}\cdot\partial_\beta\mathbf{h})\left\{\mu i q_\alpha\delta_{\beta\gamma} + \frac{\lambda}{2}iq_\gamma\delta_{\alpha\beta}\right\}u^\gamma_\mathbf{q}.
\end{aligned}$$

Gaussian integration For each momenta $\mathbf{q} \neq 0$ integration over $u^\alpha_\mathbf{q}$ has the form [2]

$$\int \prod_{\mu,\nu} \frac{(du^\mu_\mathbf{q})^* \wedge du^\nu_\mathbf{q}}{2\pi i} \exp\left\{-\frac{1}{2T}(u^\alpha_\mathbf{q})^* A^\mathbf{q}_{\alpha\beta} u^\beta_\mathbf{q} + \frac{1}{2T}(B^\mathbf{q}_\gamma)^* u^\gamma_\mathbf{q} + \frac{1}{2T} B^\mathbf{q}_\gamma (u^\gamma_\mathbf{q})^*\right\} =$$
$$= \exp\left\{\frac{1}{2T}(B^\mathbf{q}_\gamma)^* (A^\mathbf{q})^{-1}_{\gamma\gamma'} B^\mathbf{q}_{\gamma'}\right\} \frac{T^D}{\det A^\mathbf{q}},$$

with the following matrix and vector

$$\begin{aligned} A^\mathbf{q}_{\alpha\beta} &= \mu q^2 \delta_{\alpha\beta} + (\mu+\lambda) q_\alpha q_\beta, \\ B^\mathbf{q}_\gamma &= \frac{i}{2}\int d^D\mathbf{x}\, e^{-i\mathbf{q}\mathbf{x}} (\partial_\alpha \mathbf{h} \cdot \partial_\beta \mathbf{h}) \left\{\mu q_\alpha \delta_{\beta\gamma} + \mu q_\beta \delta_{\alpha\gamma} + \lambda q_\gamma \delta_{\alpha\beta}\right\}, \end{aligned}$$

Since det is simply a number (not a function of h) it may be omitted in $\mathcal{F}_\text{eff}$, thus $B^+ A^{-1} B$ is of interest. Inversion of the 2×2 matrix $A^\mathbf{q}$ is easily done in the basis of $\hat{\mathbf{q}} = \mathbf{q}/q$ and its orthogonal partner $\hat{\mathbf{q}}_\perp$ since matrix is diagonal in it [3].

$$\begin{aligned} A^\mathbf{q}_{\alpha\beta} &= \mu q^2 \delta_{\alpha\beta} + (\mu+\lambda) q_\alpha q_\beta = q^2\left(\mu P^\perp_{\alpha\beta} + (2\mu+\lambda) P^\parallel_{\alpha\beta}\right) \\ (A^\mathbf{q})^{-1}_{\alpha\beta} &= \frac{1}{q^2}\left(\frac{1}{\mu} P^\perp_{\alpha\beta} + \frac{1}{2\mu+\lambda} P^\parallel_{\alpha\beta}\right). \end{aligned}$$

Here $P^\parallel_{\alpha\beta} = q_\alpha q_\beta / q^2$ is projector to the $\mathbf{q}$ line and $P^\perp_{\alpha\beta} = \delta_{\alpha\beta} - q_\alpha q_\beta/q^2$ is projector to its orthogonal complement.

Altogether, effective free energy constitutes of the following contributions

$$\begin{aligned} \mathcal{F}_\text{eff} &= \mathcal{F}_h - \frac{1}{2}(B^\mathbf{q}_\gamma)^* (A^\mathbf{q})^{-1}_{\gamma\gamma'} B^\mathbf{q}_{\gamma'} \\ &= \frac{1}{2}\int d^D\mathbf{x} \left\{\varkappa(\nabla^2\mathbf{h})^2 + M_{\alpha\beta}\left(\varepsilon_\alpha + \frac{\overline{K}_\alpha}{2}\right)\left(\varepsilon_\beta + \frac{\overline{K}_\beta}{2}\right)\right\} + \delta\mathcal{F}^{\mathbf{q}=0}_\text{eff} + \delta\mathcal{F}^{\mathbf{q}\neq 0}_\text{eff}. \end{aligned}$$

where I singled out the interaction with $\mathbf{q} \neq 0$ coming from integration over $\mathbf{u}$ and $\frac{\mu}{2}hh$, $\frac{\lambda}{4}hh$ terms and interaction on $\mathbf{q} = 0$ produced by $\overline{KK}$ and $\frac{\mu}{2}hh$, $\frac{\lambda}{4}hh$ terms in $\mathcal{F}_h$.

Momentum dependent interaction

$$\begin{aligned} \delta\mathcal{F}^{\mathbf{q}\neq 0}_\text{eff} &= \frac{1}{8}\int' (d\mathbf{q}) \int (d\mathbf{k} d\mathbf{k}') k_\alpha (-k-q)_\beta (\mathbf{h}_\mathbf{k} \cdot \mathbf{h}_{-\mathbf{k}-\mathbf{q}}) k'_{\alpha'} (-k'+q)_{\beta'} (\mathbf{h}_{\mathbf{k}'} \cdot \mathbf{h}_{-\mathbf{k}'+\mathbf{q}}) \cdot C^\mathbf{q}_{\alpha\beta\alpha'\beta'}, \\ &= \frac{1}{8}\int' (d\mathbf{q}) \int d^D\mathbf{x}\, d^D\mathbf{x}' \left.(\partial_\alpha \mathbf{h} \cdot \partial_\beta \mathbf{h})\right|_\mathbf{x} \cdot \left.(\partial_{\alpha'} \mathbf{h} \cdot \partial_{\beta'} \mathbf{h})\right|_{\mathbf{x}'} \cdot e^{-i\mathbf{q}\mathbf{x}}\, C_{\alpha\beta\alpha'\beta'}(-i\partial_{\mathbf{x}'})\, e^{i\mathbf{q}\mathbf{x}'}. \end{aligned} \quad (2.9)$$

[2] For $\mathbf{q} = 0$ coupling is absent and therefore $B^+ A^{-1} B$ term is only contributes to the $\mathcal{F}_\text{eff}$ for $\mathbf{q} \neq 0$.
[3] Here I explicitly used two dimensionality of the setup, so that further on $D = 2$.

From the line (2.9) it is clear that $C_{\alpha\beta\alpha'\beta'}$ is contracted with tensor symmetric in α, β and α', β' indices, thus, $\alpha \leftrightarrow \beta$ and $\alpha' \leftrightarrow \beta'$ permutations are allowed in expressions below.

$$C^{\mathbf{q}}_{\alpha\beta\alpha'\beta'} = 2\mu\delta_{\alpha\alpha'}\delta_{\beta\beta'} + \lambda\delta_{\alpha\beta}\delta_{\alpha'\beta'} - \frac{2\mu q_\alpha\delta_{\beta\gamma} + \lambda q_\gamma\delta_{\alpha\beta}}{q}\left(\frac{1}{\mu}P^{\perp}_{\gamma\gamma'} + \frac{1}{2\mu+\lambda}P^{\parallel}_{\gamma\gamma'}\right)\frac{2\mu q_{\alpha'}\delta_{\beta'\gamma'} + \lambda q_{\gamma'}\delta_{\alpha'\beta'}}{q}$$

$$= \underbrace{2\mu\delta_{\alpha\alpha'}\delta_{\beta\beta'} - 4\mu P^{\parallel}_{\alpha\alpha'}P^{\perp}_{\beta\beta'} - \frac{4\mu^2}{2\mu+\lambda}P^{\parallel}_{\alpha\alpha'}P^{\parallel}_{\beta\beta'}}_{\text{tensor structure: } \#_{\alpha\alpha'}\#_{\beta\beta'}} + \underbrace{\frac{2\mu\lambda}{2\mu+\lambda}\left[\delta_{\alpha\beta}\delta_{\alpha'\beta'} - P^{\parallel}_{\alpha\beta}\delta_{\alpha'\beta'} - \delta_{\alpha\beta}P^{\parallel}_{\alpha'\beta'}\right]}_{\text{tensor structure: } \#_{\alpha\beta}\#_{\alpha'\beta'}}$$

Here I substitute $\delta_{\alpha\beta} = P^{\parallel}_{\alpha\beta} + P^{\perp}_{\alpha\beta}$ everywhere and use that $P^{\parallel}_{\alpha\alpha'}P^{\parallel}_{\beta\beta'} = q^{-4}q_\alpha q_\beta q_{\alpha'}q_{\beta'} = P^{\parallel}_{\alpha\beta}P^{\parallel}_{\alpha'\beta'}$.

$$= \underbrace{2\mu P^{\perp}_{\alpha\alpha'}P^{\perp}_{\beta\beta'} + \cancel{\frac{2\mu\lambda}{2\mu+\lambda}P^{\parallel}_{\alpha\alpha'}P^{\parallel}_{\beta\beta'}}}_{\text{tensor structure: } \#_{\alpha\alpha'}\#_{\beta\beta'}} + \underbrace{\frac{2\mu\lambda}{2\mu+\lambda}\left[P^{\perp}_{\alpha\beta}P^{\perp}_{\alpha'\beta'} - \cancel{P^{\parallel}_{\alpha\beta}P^{\parallel}_{\alpha'\beta'}}\right]}_{\text{tensor structure: } \#_{\alpha\beta}\#_{\alpha'\beta'}}$$

$$= 2\mu P^{\perp}_{\alpha\alpha'}P^{\perp}_{\beta\beta'} + \frac{2\mu\lambda}{2\mu+\lambda}P^{\perp}_{\alpha\beta}P^{\perp}_{\alpha'\beta'}. \tag{2.10}$$

For $D = 2$ this expression may be simplified further owning to the existence of fully anti–symmetric tensor $\epsilon_{\alpha\beta}$ that allows to present $P^{\perp}_{\alpha\beta} = p^{-2}p_\alpha p_\beta$ where $p_\alpha = \epsilon_{\alpha\beta}q_\beta$ is vector perpendicular to $\mathbf{q} \perp \mathbf{p}$.

$$\frac{p_\alpha p_\beta}{p^2} = \frac{\epsilon_{\alpha\alpha'}\epsilon_{\beta\beta'}q_{\alpha'}q_{\beta'}}{p_\gamma p_\gamma} = \frac{(\delta_{\alpha\beta}\delta_{\alpha'\beta'} - \delta_{\alpha'\beta}\delta_{\alpha\beta'})q_{\alpha'}q_{\beta'}}{\epsilon_{\gamma\mu}\epsilon_{\gamma\nu}q_\mu q_\nu} = \frac{\delta_{\alpha\beta}q^2 - q_\alpha q_\beta}{q^2} = P^{\perp}_{\alpha\beta}$$

Therefore, $P^{\perp}_{\alpha\alpha'}P^{\perp}_{\beta\beta'} = p^{-4}p_\alpha p_\beta p_{\alpha'}p_{\beta'} = P^{\perp}_{\alpha\beta}P^{\perp}_{\alpha'\beta'}$ In other words, only for $D = 2$ operator $P^{\perp}$ may be presented as tensor product of two vectors.

$$P^{\perp}_{D=2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad P^{\perp}_{D=3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \neq \mathbf{v} \otimes \mathbf{w}$$

All in all, for $D = 2$ dimensional case expression (2.10) becomes

$$C^{\mathbf{q}}_{\alpha\beta\alpha'\beta'} = \frac{4\mu(\mu+\lambda)}{2\mu+\lambda}P^{\perp}_{\alpha\beta}P^{\perp}_{\alpha'\beta'} = Y_0 P^{\perp}_{\alpha\beta}P^{\perp}_{\alpha'\beta'}, \qquad Y_0 = \frac{2\mu(2\mu+D\lambda)}{2\mu+\lambda(D-1)},$$

where Y_0 is Young modulus of two dimensional membrane. And since in 2D $\left|\hat{P}^{\perp}_{\mathbf{q}}\mathbf{k}\right| = |[\mathbf{k} \times \hat{\mathbf{q}}]|$, with unit vector $\hat{\mathbf{q}} = \mathbf{q}/q$, contribution $\delta\mathcal{F}_{\text{eff}}^{\mathbf{q}\neq 0}$ can be rewritten as

$$\delta\mathcal{F}_{\text{eff}}^{\mathbf{q}\neq 0} = \frac{Y_0}{8}\int'(d\mathbf{q})\int(d\mathbf{k}d\mathbf{k}')[\mathbf{k}'\times\hat{\mathbf{q}}]^2[\mathbf{k}\times\hat{\mathbf{q}}]^2(\mathbf{h}_{\mathbf{k}'}\cdot\mathbf{h}_{-\mathbf{k}'+\mathbf{q}})(\mathbf{h}_{\mathbf{k}}\cdot\mathbf{h}_{-\mathbf{k}-\mathbf{q}}).$$

Momentum independent interaction Now let me get back to the $\mathbf{q} = 0$ contribution. Remembering how to come from integration to summation over discrete impulses (2.8) I modify the following expressions

$$\frac{1}{2}\int d^D\mathbf{x}\left\{\frac{\mu}{2}(\partial_\alpha\mathbf{h}\cdot\partial_\beta\mathbf{h})^2 + \frac{\lambda}{4}(\partial_\alpha\mathbf{h}\cdot\partial_\alpha\mathbf{h})^2\right\} =$$
$$= \frac{1}{2}\frac{1}{4L^D}\int(d\mathbf{k}d\mathbf{k}')\left[2\mu(\mathbf{k}\cdot\mathbf{k}')^2 + \lambda(kk')^2\right](\mathbf{h}_{\mathbf{k}}\cdot\mathbf{h}_{-\mathbf{k}})(\mathbf{h}_{\mathbf{k}'}\cdot\mathbf{h}_{-\mathbf{k}'}) + \int'(d\mathbf{q})\ldots$$

Space averaged anomalous deformations transformed as

$$\overline{K}_\alpha = \int d^D\mathbf{x}\,(\partial_\alpha\mathbf{h}\cdot\partial_{(\alpha)}\mathbf{h}) = \frac{1}{L^D}\int(d\mathbf{k})k_\alpha k_{(\alpha)}(\mathbf{h}_{\mathbf{k}}\cdot\mathbf{h}_{-\mathbf{k}}),$$

$$\frac{1}{2}\int d^D\mathbf{x}\,M_{\alpha\beta}\overline{K}_\alpha\overline{K}_\beta = \frac{1}{2}\frac{1}{L^D}\left[2\mu(k_x^2(k_x')^2 + k_y^2(k_y')^2) + \lambda(kk')^2\right](\mathbf{h}_{\mathbf{k}}\cdot\mathbf{h}_{-\mathbf{k}})(\mathbf{h}_{\mathbf{k}'}\cdot\mathbf{h}_{-\mathbf{k}'})$$

Sum of the two contributions gives

$$\delta\mathcal{F}_{\text{eff}}^{\mathbf{q}=0} = \frac{\mu}{2L^D}\int(d\mathbf{k}d\mathbf{k}')k_xk_yk_x'k_y'(\mathbf{h}_{\mathbf{k}}\cdot\mathbf{h}_{-\mathbf{k}})(\mathbf{h}_{\mathbf{k}'}\cdot\mathbf{h}_{-\mathbf{k}'}) = \frac{\mu}{2L^D}\left[\int d^D\mathbf{x}\,(\partial_x\mathbf{h}\cdot\partial_y\mathbf{h})\right]^2.$$

Effective action Altogether, effective free energy depending only on out–of–plane fluctuations reads (compare with [7, (19)])

$$\mathcal{F}_{\text{eff}}[\mathbf{h}] = \frac{1}{2}\int d^D\mathbf{x}\left\{\varkappa(\nabla^2\mathbf{h})^2 + M_{\alpha\beta}\left(\varepsilon_\alpha + \frac{\overline{K}_\alpha}{2}\right)\left(\varepsilon_\beta + \frac{\overline{K}_\beta}{2}\right)\right\} + \frac{\mu}{2L^D}\left[\int d^D\mathbf{x}\,(\partial_x\mathbf{h}\cdot\partial_y\mathbf{h})\right]^2 +$$
$$+ \frac{Y_0}{8}\int(d\mathbf{q}d\mathbf{k}d\mathbf{k}')\frac{[\mathbf{k}\times\mathbf{q}]^2[\mathbf{k}'\times\mathbf{q}]^2}{q^4}(\mathbf{h}_{\mathbf{k}}\cdot\mathbf{h}_{-\mathbf{k}-\mathbf{q}})(\mathbf{h}_{\mathbf{k}'}\cdot\mathbf{h}_{-\mathbf{k}'+\mathbf{q}}).$$

2.1.4 Balance equation

Constriction (2.5) has the physical meaning of balance equation and stretching factors ε are thermodynamic variables that play the role of the volume.

$$\sigma_\alpha = \frac{\partial f}{\partial \varepsilon_\alpha}, \quad f = \frac{F}{L^2} = -\frac{T}{L^2} \ln \int D[\mathbf{h}] e^{-\mathcal{F}_{\text{eff}}[\mathbf{h},\varepsilon]/T}.$$

If strain tensor was linearized $u_{\alpha\beta} = \frac{1}{2}(\partial_\alpha u_\beta + \partial_\beta u_\alpha)$ then as it is seen from (2.6), equation of balance would simply read

$$\sigma^0_\alpha = M_{\alpha\beta}\varepsilon_\beta$$

with matrix $M_{\alpha\beta}$ given by (2.7). Presence of the h^2 term in strain tensor leads to non–linear equation of balance [8].

$$\sigma_\alpha = M_{\alpha\beta}\left(\varepsilon_\beta + \left\langle \overline{K}_\beta \right\rangle\right), \qquad \left\langle \overline{K}_\beta \right\rangle(\varepsilon) = \frac{\int D[\mathbf{h}] \overline{K}_\beta e^{-\mathcal{F}_{\text{eff}}[\mathbf{h},\varepsilon]/T}}{\int D[\mathbf{h}] e^{-\mathcal{F}_{\text{eff}}[\mathbf{h},\varepsilon]/T}}, \tag{2.11}$$

This equation may be inverted and $\varepsilon = \varepsilon(\sigma)$ dependence obtained.

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \end{pmatrix} = \frac{1}{Y_0}\begin{pmatrix} 1 & -\nu_0 \\ -\nu_0 & 1 \end{pmatrix}\begin{pmatrix} \sigma_x \\ \sigma_y \end{pmatrix} - \frac{1}{2}\begin{pmatrix} \left\langle \overline{K}_x \right\rangle(\sigma) \\ \left\langle \overline{K}_y \right\rangle(\sigma) \end{pmatrix} \tag{2.12}$$

where Y_0 and ν_0 are classical expressions for Young's modulus and Poisson ratio respectively. Because of the non–linearity, equation (2.11) may have non–zero deformation ε even in the absence of stress $\sigma = 0$, then true deformations would be given by $\varepsilon(\sigma) - \varepsilon(0)$.

Absolute Poisson ratio Let me now consider a membrane subjected to an uniaxial stress in the x direction $\sigma_x = \sigma$, $\sigma_y = 0$. The balance equations (2.12) become simpler and I extract expression for PR.

$$\nu = -\frac{\varepsilon_y}{\varepsilon_x} = \frac{\nu_0 + Y_0\left\langle \overline{K}_y \right\rangle(\sigma,0)/2\sigma}{1 - Y_0\left\langle \overline{K}_x \right\rangle(\sigma,0)/2\sigma} \approx -\frac{\left\langle \overline{K}_y \right\rangle(\sigma,0)}{\left\langle \overline{K}_x \right\rangle(\sigma,0)}.$$

Last approximation is done in the limit of large anomalous deformations, $\sigma \ll Y_0\left\langle \bar{\zeta}_\alpha \right\rangle \sim Y_0 T/\varkappa$.

Differential Poisson ratio For the finite stress $\sigma_x = \sigma + \delta\sigma$, $\sigma_y = \sigma$ linear responses $\delta\varepsilon_y = -\nu^{\text{diff}}\delta\varepsilon_x$ are connected via, so called, differential Poisson ratio. From (2.12) follows

$$\nu = -\frac{\varepsilon_y}{\varepsilon_x} = \left.\frac{\nu_0 + \frac{Y_0}{2}\left(\frac{\partial\left\langle \overline{K}_y \right\rangle}{\partial\sigma_x}\right)_{\sigma_y}}{1 - \frac{Y_0}{2}\left(\frac{\left\langle \overline{K}_x \right\rangle}{\partial\sigma_x}\right)_{\sigma_y}}\right|_{\sigma_x=\sigma_y=\sigma} \approx -\frac{\left(\partial\left\langle \overline{K}_y \right\rangle/\partial\sigma_x\right)_{\sigma_y}}{\left(\partial\left\langle \overline{K}_x \right\rangle/\partial\sigma_x\right)_{\sigma_y}}.$$

Chapter 3

Perturbation theory

Effective action for flexural phonon derived in previous chapter can be split into three parts.

$$
\begin{aligned}
\mathcal{F}^{(2)}_{\text{eff}}[\mathbf{h}] &= \frac{1}{2}\int (d\mathbf{p})\left[\varkappa p^4 + \varepsilon_\alpha M_{\alpha\beta} p^2_{(\beta)}\right](\mathbf{h}_\mathbf{p}\cdot\mathbf{h}_{-\mathbf{p}}),\\
\mathcal{F}^{\mathbf{q}=0}_{\text{eff}}[\mathbf{h}] &= \frac{1}{4}\int (d\mathbf{k}d\mathbf{k}')\frac{1}{L^D}\left[k^2_{(\alpha)}\frac{M_{\alpha\beta}}{2}k'^2_{(\beta)} + 2\mu k_1k_2k'_1k'_2\right](\mathbf{h}_\mathbf{k}\cdot\mathbf{h}_{-\mathbf{k}})(\mathbf{h}_{\mathbf{k}'}\cdot\mathbf{h}_{-\mathbf{k}'})\\
\mathcal{F}^{\mathbf{q}\neq0}_{\text{eff}}[\mathbf{h}] &= \frac{Y_0}{8}\int (d\mathbf{k}d\mathbf{k}')\int'(d\mathbf{q})[\mathbf{k}\times\hat{\mathbf{q}}]^2[\mathbf{k}'\times\hat{\mathbf{q}}]^2(\mathbf{h}_\mathbf{k}\cdot\mathbf{h}_{-\mathbf{k}-\mathbf{q}})(\mathbf{h}_{\mathbf{k}'}\cdot\mathbf{h}_{-\mathbf{k}'+\mathbf{q}})
\end{aligned}
$$

As it is evident from Subsection 2.1.4 all information about Hooke's law is contained in correlation function (I drop brackets and bar sign from now on)

$$
K_\alpha = \int \frac{d^D\mathbf{x}}{L^D}\left\langle(\partial_\alpha\mathbf{h}\cdot\partial_{(\alpha)}\mathbf{h})\right\rangle = \int\frac{(d\mathbf{p})}{L^D}p^2_{(\alpha)}\left\langle(\mathbf{h}_\mathbf{p}\cdot\mathbf{h}^*_\mathbf{p})\right\rangle = d_c\int(d\mathbf{p})p^2_{(\alpha)}\mathcal{G}_\mathbf{p},
$$

where $\mathcal{G}_\mathbf{p}$ is exact Green's function. Before developing a perturbation theory I take a deeper look at interaction structure.

3.1 Diagrammatics

Bare Green's function is given by

$$
\begin{aligned}
\left\langle h^\mu_\mathbf{p} h^\nu_{-\mathbf{q}}\right\rangle_0 &= \int\left[\prod_{\mathbf{k},\alpha\beta}\frac{dh^\alpha_\mathbf{k}\wedge(dh^\beta_\mathbf{k})^*}{2\pi i(TL^D)^{d_c}}\right]h^\mu_\mathbf{p}(h^\nu_\mathbf{q})^*\exp\left[-\frac{1}{2TL^D}\sum_\mathbf{p}\left[\varkappa p^4+\sigma^0_\beta p^2_{(\beta)}\right](\mathbf{h}_\mathbf{p}\cdot\mathbf{h}^*_\mathbf{p})\right]\\
&= (2\pi)^D\delta(\mathbf{p}-\mathbf{q})\delta^{\mu\nu}G^0_\mathbf{p}, \qquad G^0_\mathbf{p} = \frac{T}{\varkappa p^4+\sigma^0_1p^2_1+\sigma^0_2p^2_2}.
\end{aligned}
$$

I have two distinct types of interaction presented on Figure 5.

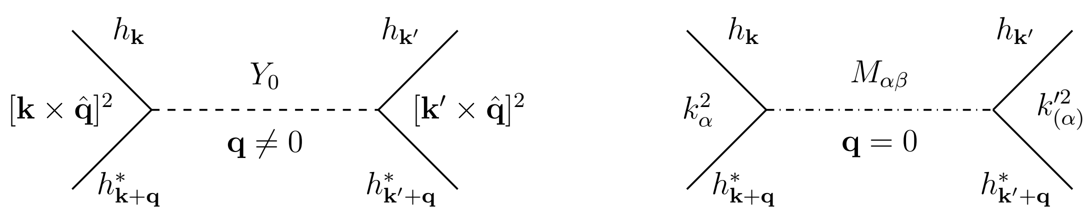


Figure 5: Interaction diagrams.

Because of the momentum constriction tadpole–like diagrams are only allowed for momenta non–bearing interactions. That is why there is only self–energy like contribution in first order in Y_0.

$$
\begin{aligned}
\mathcal{G}_{\mathbf{p}}+ &= -\frac{Y_0}{T}\int (d\mathbf{k}d\mathbf{k}'d'\mathbf{q})[\mathbf{k}\times\hat{\mathbf{q}}]^2[\mathbf{k}'\times\hat{\mathbf{q}}]^2\left\langle(\mathbf{h_p}\cdot\mathbf{h}^*_{\mathbf{p}})(\mathbf{h_k}\cdot\mathbf{h}^*_{\mathbf{k+q}})(\mathbf{h}_{\mathbf{k}'}\cdot\mathbf{h}^*_{\mathbf{k}'-\mathbf{q}})\right\rangle_0 \\
&= -\frac{d_cY_0}{T}\int'(d\mathbf{q})[\mathbf{p}\times\hat{\mathbf{q}}]^4(2\pi)^D\delta(\mathbf{k}=0)[G^0_{\mathbf{p}}]^2G^0_{\mathbf{p-q}} \\
&= -\frac{d_cY_0}{T}[G^0_{\mathbf{p}}]^2\int'(d\mathbf{q})[\mathbf{p}\times\hat{\mathbf{q}}]^4G^0_{\mathbf{p-q}}.
\end{aligned}
$$

Another correction, of the same order in L^{-D}, comes from $\mathcal{F}^{\mathbf{q}=0}_{\text{eff}}$ term with $M_{\alpha\beta}$ and the following coupling.

$$
\begin{aligned}
\mathcal{G}_{\mathbf{p}}+ &= -\frac{1}{T}\int\frac{(d\mathbf{k}d\mathbf{k}')}{L^D}k^2_{(\alpha)}M_{\alpha\beta}(k'_{(\beta)})^2\left\langle(\mathbf{h_p}\cdot\mathbf{h}^*_{\mathbf{p}})(\mathbf{h_k}\cdot\mathbf{h}^*_{\mathbf{k}})(\mathbf{h}_{\mathbf{k}'}\cdot\mathbf{h}^*_{\mathbf{k}'})\right\rangle \\
&= -\frac{d_c^2}{T}p^2_{(\alpha)}M_{\alpha\beta}\int\frac{(d\mathbf{k}')}{L^D}(k'_{(\beta)})^2(2\pi)^D\delta(\mathbf{k}=0)(2\pi)^D\delta(\mathbf{k}'=0)[G^0_{\mathbf{p}}]^2G^0_{\mathbf{k}'} \\
&= -\frac{d_c^2}{T}p^2_{(\alpha)}M_{\alpha\beta}[G^0_{\mathbf{p}}]^2\int(d\mathbf{k}')(k'_{(\beta)})^2G^0_{\mathbf{k}'}.
\end{aligned}
$$

Tadpole diagram with $2\mu k_1k_2k'_1k'_2$ is zero because of its tensor structure. Diagrams corresponding to these two contributions are depicted on Figure 6.

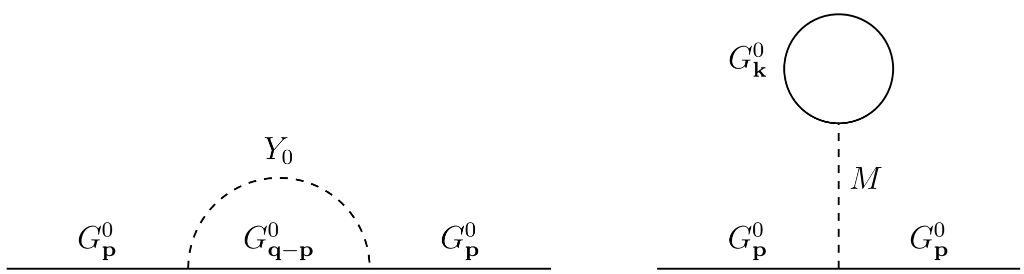


Figure 6: First–oder corrections (in interaction strength).

The problem with the former expressions is that σ^0_1, σ^0_2 are some unknown numbers here, they are connected with true tension $\sigma_{1,2}$ through (2.11). In other words, I would like to operate with

$$G_{\mathbf{p}} = \frac{T}{\varkappa p^4+\sigma_1p_1^2+\sigma_2p_2^2}$$

instead of $G^0_{\mathbf{p}}$. It so happens that's exactly what momenta–free interaction $\mathcal{F}_{\mathbf{q}=0}$ is responsible for. One way to see that is to use Ward's identity [7], but simpler way is to isolate all contributions in leading order in L^{-D}.

Infinite size L limit

Any diagram that does not vanish in $L \to +\infty$ limit has the form of the diagram that one can produce using only momenta–bearing interaction Y_0 with bare Green function $G^0_{\mathbf{p}}$ replaced by the $G_{\mathbf{p}}$. Function $G_{\mathbf{p}}$ is given by the series re–summed on Figure 7. It so happens that

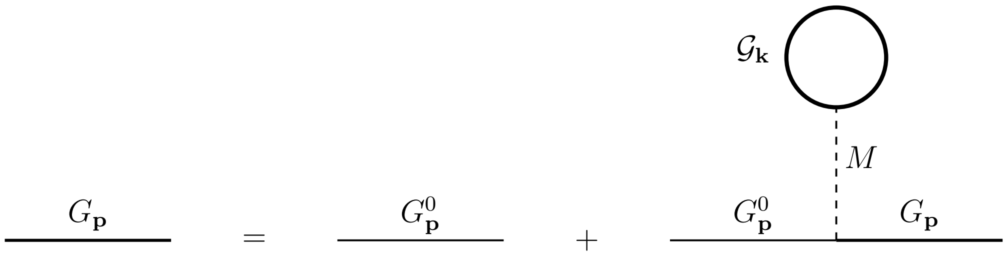


Figure 7: Stress renormalization equation.

renormalization of σ provided by $M_{\alpha\beta}$ changes $\sigma^0 \mapsto \sigma$ exactly. Diagrammatic equation on Fig. Figure 7 reads

$$G_{\mathbf{p}} = G^0_{\mathbf{p}} - G^0_{\mathbf{p}} G_{\mathbf{p}} p^2_{(\alpha)} \frac{M_{\alpha\beta}}{2T} \int (d\mathbf{k}) k^2_{(\beta)} d_c \mathcal{G}_{\mathbf{k}}.$$

and results in correction of σ, namely,

$$G_{\mathbf{p}} = \left((G^0_{\mathbf{p}})^{-1} + \delta\sigma_\alpha p^2_{(\alpha)}\right)^{-1}, \qquad \delta\sigma_\alpha = \frac{M_{\alpha\beta}}{2} \int (d\mathbf{k}) k^2_{(\beta)} d_c \mathcal{G}_{\mathbf{k}} = \frac{1}{2} M_{\alpha\beta} \left\langle \bar{K}_\beta \right\rangle.$$

That being said, I exclude interaction with $\mathbf{q} = 0$ completely. In other words, from now on I work with the following expression for free energy.

$$\mathcal{F}^{(2)}_{\text{eff}}[\mathbf{h}] = \frac{1}{2} \int (d\mathbf{p}) \left[\varkappa p^4 + \sigma_\beta p^2_{(\beta)}\right] (\mathbf{h}_{\mathbf{p}} \cdot \mathbf{h}_{-\mathbf{p}}), \tag{3.1}$$

$$\mathcal{F}_{\text{eff}}[\mathbf{h}] = \frac{Y_0}{8} \int (d\mathbf{k} d\mathbf{k}') \int_{\mathbf{q} \neq 0} (d\mathbf{q}) [\mathbf{k} \times \hat{\mathbf{q}}]^2 [\mathbf{k}' \times \hat{\mathbf{q}}]^2 (\mathbf{h}_{\mathbf{k}} \cdot \mathbf{h}_{-\mathbf{k}-\mathbf{q}})(\mathbf{h}_{\mathbf{k}'} \cdot \mathbf{h}_{-\mathbf{k}'+\mathbf{q}})$$

Green function from now one stands for

$$G_{\mathbf{p}} = \frac{T}{\varkappa p^4 + \sigma_1 p_1^2 + \sigma_2 p_2^2}.$$

and the only Y–interaction vertex present on Figure 5 is left.

High d_c expansion

Now I develop perturbation theory. Since initial action does not contain small parameter in fort of unharmonic term I construct it myself by treating $\mathbf{h}$ as a vector of size d_c. Each Green function function bubble of proportional to d_c, so I have to encounter screening of $\mathcal{F}_{\text{int}}$ via polarization bubble.

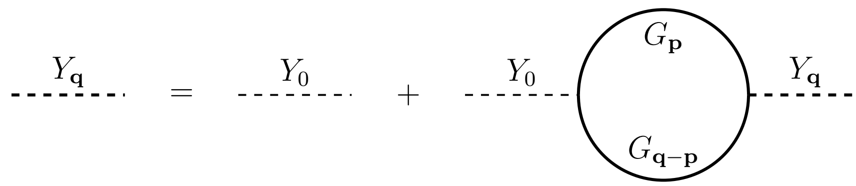


Figure 8: Screening of interaction.

Interaction constant effectively starts bearing momentum $Y_{\mathbf{q}}$ and given by

$$\frac{Y_{\mathbf{q}}}{2} = \frac{Y_0}{2}\left(1 + \frac{Y_0}{2}\Pi_{\mathbf{q}}\right)^{-1}$$

where $\Pi_{\mathbf{q}}$ is polarization operator given by

$$\Pi_{\mathbf{q}} = \frac{d_c}{T}\int (d\mathbf{k}) G_{\mathbf{k}} G_{\mathbf{q}-\mathbf{k}} [\mathbf{k}\times\hat{\mathbf{q}}]^4 = \frac{d_c T}{\varkappa\sigma} P\left(\frac{\mathbf{q}}{q_0}\right)$$

$$P(\mathbf{q}) = \int \frac{(d\mathbf{k})}{k^4 + k_1^2}\frac{[\mathbf{k}\times\hat{\mathbf{q}}]^4}{(\mathbf{k}-\mathbf{q})^4 + (k_1 - q_1)^2}.$$

Here $q_0 = \sqrt{\sigma/\varkappa}$ is characteristic scale set by stress and $P(\mathbf{q})$ is dimensionless Polarization operator that is calculated in Appendix A.2. At the moment it is only important how P behaves at large argument values.

$$P(q \gg 1) = \int \frac{(d\mathbf{k})[\mathbf{k}\times\hat{\mathbf{q}}]^4}{k^4(\mathbf{k}-\mathbf{q})^4} = \frac{3}{16\pi}\frac{1}{q^2}.$$

From here new characteristic scale set by interaction strength q_* appears, that tends to infinity together with d_c. Since all the anomalous physics comes from $q \ll q_*$ range, such artificial parameter expected to give good approximation.

$$\frac{Y_{\mathbf{q}}}{2} \approx \frac{1}{\Pi_{\mathbf{q}}} = \frac{\varkappa\sigma}{d_c T}\frac{1}{P(\mathbf{q}/q_0)}, \qquad \frac{\sigma}{\varkappa} \equiv q_0^2 \ll q^2 \ll q_*^2 \equiv \frac{3}{16\pi}\frac{d_c Y_0 T}{\varkappa^2},$$

After the screening is accounted for, $Y_{\mathbf{q}}$ becomes explicitly small in $1/d_c \to 0$. To draw all the diagrams in n–th order one has to count number of loops (closed Green's function lines) — each gives d_c and numbers of interaction lines — each gives $1/d_c$.

3.2 Scaling Ansatz

Here I for a moment forget about perturbation theory and describe general statement that is known about exact Green function. Formally, exact Green function is given by sum of asymptotic perturbation series, which can be formally rewritten as the series for exact self–energy $\mathbf{\Sigma_p}$ and exact polarization operator $\mathbf{\Pi_q}$.

$$\mathcal{G}_{\mathbf{p}} = \left(G_{\mathbf{p}}^{-1} + T\mathbf{\Sigma}_{\mathbf{p}}\right)^{-1} \qquad\qquad \frac{\mathcal{Y}_{\mathbf{q}}}{2} = \frac{Y_0}{2}\left(1 + \mathbf{\Pi}_{\mathbf{q}}\frac{Y_0}{2}\right)^{-1}. \tag{3.2}$$

In diagrams below solid line denotes exact Green function $\mathcal{G}_{\mathbf{p}}$ and dashed line stands for $\mathcal{Y}_{\mathbf{q}}$ — interaction screened with exact polarization operator. There is an ansatz for exact Green

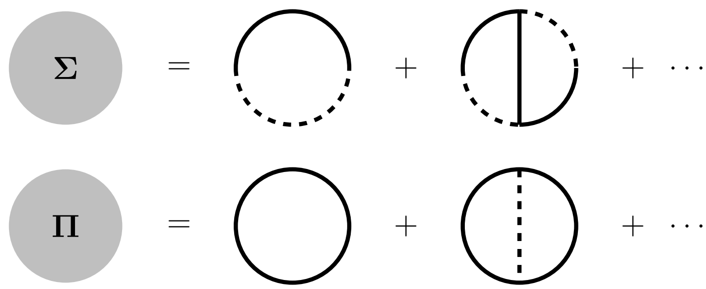


Figure 9: Formal asymptotic series for self–energy and polarization operator.

function

$$\mathcal{G}_{\mathbf{p}} = \frac{T}{\varkappa_{\mathbf{p}} p^4 + \sigma_1 p_1^2 + \sigma_2 p_2^2}, \qquad \varkappa_{\mathbf{p}} = \varkappa \begin{cases} 1, & p \gg p_*, \\ (p_*/p)^{\eta}, & p \ll p_*. \end{cases} \tag{3.3}$$

That asymptotically satisfies equations (3.2) since scaling of bending rigidity implies the same scaling for any diagram in series presented on Figure 9.

$$\Sigma_{\mathbf{p}} \propto \frac{\varkappa}{d_c T}\left(\frac{p_*}{p}\right)^{\eta} p^4, \quad \Pi_{\mathbf{q}} \propto \frac{d_c T}{\varkappa^2 q^2}\left(\frac{q}{p_*}\right)^{2\eta}, \qquad p, q \ll p_* = \sqrt{\frac{3A}{16\pi}\frac{d_c Y_0 T}{\varkappa^2}}.$$

Here p_* coincides with the scale set by interaction strength defined in previous section up to an unknown prefactor A of order unity and $\eta > 0$ is unknown critical index. Number η is unambiguously connected with exponent (1.1) from anomalous Hooke's law

$$\alpha = \frac{\eta}{2-\eta}$$

as will be shown later, so it is of great interest to find that number.

3.2.1 Self–consistent screening approximation

Self–consistent screening approximation (SCSA) allows [9] to find η in some manner, it has no controllable parameter, yet believed to give numerically satisfactory results [5]. The idea of approximation is to take only the first terms in series on Figure 9, i e. solve the following system of equations.

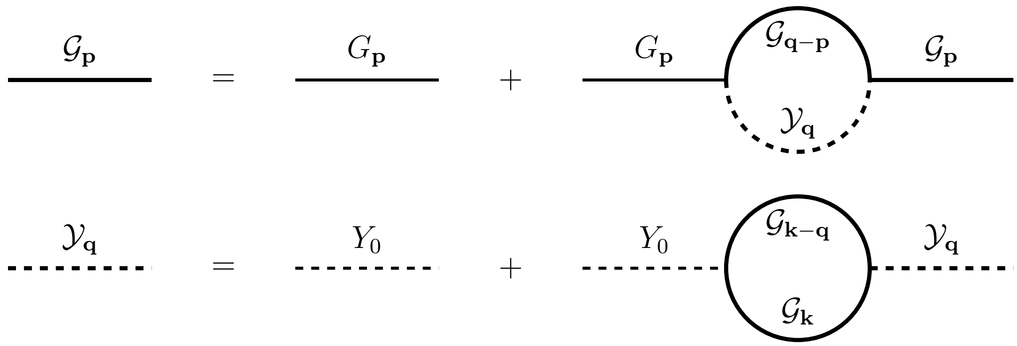


Figure 10: System of self–consistent screening approximation equations.

These diagrams (Figure 10) correspond to analytical expressions

$$\Pi_{\mathbf{q}} = \frac{d_c}{T}\int (d\mathbf{k})\mathcal{G}_{\mathbf{k}}\mathcal{G}_{\mathbf{k}-\mathbf{q}}[\mathbf{k}\times\hat{\mathbf{q}}]^4, \qquad \mathcal{G}_{\mathbf{p}} = \left(G_{\mathbf{p}}^{-1} + T\Sigma_{\mathbf{p}}\right)^{-1}$$

$$T\Sigma_{\mathbf{p}} = \int' (d\mathbf{q})\mathcal{G}_{\mathbf{p}-\mathbf{q}}Y_{\mathbf{q}}[\mathbf{p}\times\hat{\mathbf{q}}]^4, \qquad \frac{Y_{\mathbf{q}}}{2} = \frac{Y_0}{2}\left(1 + \Pi_{\mathbf{q}}\frac{Y_0}{2}\right)^{-1}.$$

In the absence of strain ($\sigma = 0$) there are two regimes presented in (3.3). For $p \gg p_*$, I assume the integral over $\mathbf{q}$ comes from $q \sim p$ and substitute $Y_{\mathbf{q}} \sim Y_0$ and see my assumption satisfied. For $p \ll p_*$, I guess power law dependence $\varkappa_{\mathbf{p}} = \varkappa(Ap_*/p)^\eta$.

$$\Pi_{\mathbf{q}} = A^{2\eta}\frac{d_c T}{\varkappa^2 p_*^{2\eta}}\Pi(\eta,\eta)q^{2\eta-2}, \qquad \Pi(\eta,\eta') = \int \frac{[\mathbf{k}\times\hat{\mathbf{q}}]^4(d\mathbf{k})}{k^{4-\eta}|\mathbf{k}-\hat{\mathbf{q}}|^{4-\eta'}}.$$

$$\Sigma_{\mathbf{p}} = \frac{2A^\eta\Sigma(\eta,\eta)}{Td_c\Pi(\eta,\eta)}\varkappa\left(\frac{p_*}{p}\right)^\eta p^4, \qquad \Sigma(\eta,\eta') = \int' \frac{[\hat{\mathbf{p}}\times\hat{\mathbf{q}}]^4 q^{2-2\eta}}{(\hat{\mathbf{p}}-\mathbf{q})^{4-\eta'}}(d\mathbf{q}).$$

Functions $\Sigma(\eta,\eta)$ and $\Pi(\eta,\eta)$ are calculated, for example, in [13, 5]. Self–consistency demands

$$\frac{2\Sigma(\eta,\eta)}{d_c\Pi(\eta,\eta)} = 1 \qquad \Rightarrow \qquad \eta = \left.\frac{4}{d_c + \sqrt{16 - 2d_c + d_c^2}}\right|_{d_c=1} \simeq 0.82 \qquad \eta \sim \frac{2}{d_c}, \;\; d_c \to \infty.$$

Number A may not be determined from that consideration.

The problem with that result, however, is that it is not done under any controllable approximation. That's why I am not satisfied with it and develop perturbation theory in $1/d_c$.

3.3 Critical exponent expansion

In order to find how critical exponent η as a function of complementary dimensionality d_c, I calculate small argument asymptotics of self–energy (without external stress $\sigma = 0$) perturbatively in $1/d_c$. Scaling ansatz (SA) tells me that at small momenta, bending rigidity behaves as

$$\frac{\varkappa_{\mathbf{p}}}{\varkappa} = 1 - \frac{T\Sigma_{\mathbf{p}}}{\varkappa p^4} \sim \left[A\frac{p_*}{p}\right]^{\eta}, \qquad p \ll p_*, \tag{3.4}$$

where exponent $\eta = \eta(d_c)$ is the number I seek to calculate up to the second order in $1/d_c$.

Averaging rules Throughout the calculations I am going to frequently use following identities

$$\left\langle \frac{[\mathbf{p}\times\mathbf{q}]^4}{(\mathbf{p}-\mathbf{q})^4} \right\rangle_{\hat{\mathbf{q}}} = \frac{3}{8}\min\{p^4, q^4\}, \tag{3.5}$$

$$\left\langle \frac{[\mathbf{p}\times\mathbf{q}]^4}{(\mathbf{p}-\mathbf{q})^2} \right\rangle_{\hat{\mathbf{q}}} = \min\{p^4, q^4\}\left(\max\{p^2, q^2\} - \frac{1}{3}\min\{p^2, q^2\}\right), \tag{3.6}$$

$$\left\langle \frac{[\mathbf{p}\times\mathbf{q}]^2}{(\mathbf{p}-\mathbf{q})^4} \right\rangle_{\hat{\mathbf{q}}} = \frac{1}{2\max\{p^2, q^2\}|p^2 - q^2|}, \tag{3.7}$$

3.3.1 First–order self–energy

Expansion of self–energy in $1/d_c$ starts from the trivial term (Figure 11–11a)

$$T\Sigma^{(1)}_{\mathbf{p}} = -\frac{Y_0 T}{\varkappa}\int \frac{[\mathbf{p}\times\hat{\mathbf{q}}]^4 (d\mathbf{q})}{(\mathbf{p}-\mathbf{q})^4(1 + \frac{Y_0}{2}\Pi_{\mathbf{q}})} = -\frac{\varkappa p_*^4}{2d_c}\mathcal{S}\left(\sqrt{2}\frac{p}{p_*}\right).$$

In the absence of stress $\sigma = 0$ polarization operator is simply

$$Y_0\Pi_{\mathbf{q}} = \frac{q_*^2}{q^2}, \qquad q_*^2 = \frac{3}{16\pi}\frac{d_c Y_0 T}{\varkappa^2},$$

and dimensionless self–energy is given by function

$$\mathcal{S}(\mathbf{p}) = \frac{16\pi}{3}\int \frac{[\mathbf{p}\times\hat{\mathbf{q}}]^4 (d\mathbf{q})}{(\mathbf{p}-\mathbf{q})^4 + (\mathbf{p}-\mathbf{q})^2} = \frac{16\pi}{3}\int \frac{[\mathbf{p}\times\mathbf{q}]^4 (d\mathbf{q})}{(\mathbf{p}-\mathbf{q})^4(q^4 + q^2)} = (3.5) =$$
$$= \frac{1}{2}\int_0^\infty \frac{dq}{q^2 + q}\min\{p^4, q^2\} = \frac{1}{2}\left\{(p^4 - 1)\ln\left(1 + p^2\right) - p^4\ln p^2 + p^2\right\}.$$

Therefore the first–oder term has the following asymptotics

$$T\Sigma^{(1)}_{\mathbf{p}} \sim \frac{2}{d_c}\varkappa p^4 \ln\left[\frac{\sqrt{2}}{e^{1/4}}\frac{p}{p_*}\right], \quad p \ll p_*$$

That allows me to find

$$A(d_c) = \frac{e^{1/4}}{\sqrt{2}} + \frac{A_1}{d_c} + \ldots, \quad \eta(d_c) = \frac{2}{d_c} + \frac{\eta_2}{d_c^2} + \ldots \tag{3.8}$$

3.3.2 Second order. SCSA–like contributions

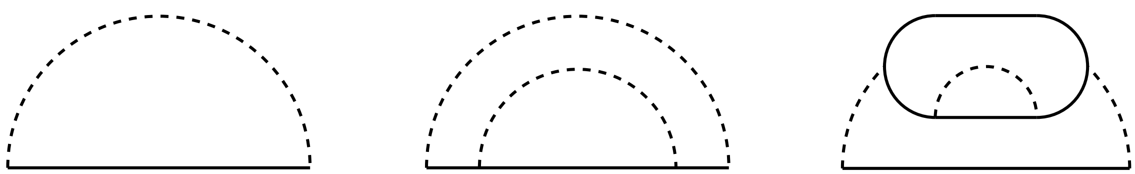
Second–order consists of two trivial SCSA–like contributions and four non–SCSA corrections. Together these two give the result that may be found from

$$\eta_{\text{SCSA}}(d_c) = \frac{4}{d_c + \sqrt{16 - 2d_c + d_c^2}} = \frac{2}{d_c} + \frac{1}{d_c^2} + \mathcal{O}\left(\frac{1}{d_c^3}\right).$$

Plugging (3.8) into (3.4) with $\eta_2 = 1 + \delta\eta_2$ and expanding in $1/d_c$ I come to

$$d_c^2 \frac{T\Sigma^{(2)}_{\mathbf{p}}}{\varkappa p^4} = -2\ln^2\left[\sqrt{2}\frac{p}{p_*}\right] + (2 + \delta\eta_2)\ln\left[\sqrt{2}\frac{p}{p_*}\right] - \frac{3}{8} - \frac{\delta\eta_2}{4} - 2A_1. \tag{3.9}$$

I see that with logarithmic accuracy SCSA–like contributions to self–energy should produce at small momenta $-2\ln^2[\sqrt{2}p/p_*] + 2\ln[p/p_*]$. Let me check that.



(11a) Fan diagram. (12a) Rainbow diagram (12b) Upset diagram

Figure 11: SCSA–like self–energy corrections up to the second order.

Rainbow diagram Explicit expression for diagram on Figure 11–11a is

$$T\Sigma^{(2a)}_{\mathbf{p}} = \frac{\varkappa p_*^4}{d_c^2}\mathcal{S}^{(2a)}\left(\sqrt{2}\frac{p}{p_*}\right), \qquad \mathcal{S}^{(2a)}(p) = \left(\frac{16\pi}{3}\right)\int \frac{(d\mathbf{q})[\mathbf{p}\times\mathbf{q}]^4 q^{-8}\mathcal{S}(q)}{(\mathbf{p}-\mathbf{q})^4 + (\mathbf{p}-\mathbf{q})^2}$$

Since I only need asymptotic at small p, I calculate

$$
\begin{aligned}
\tilde{\mathcal{S}}^{(2a)}(p) &= \left(\frac{16\pi}{3}\right)\int (d\mathbf{q})\frac{[\mathbf{p}\times\mathbf{q}]^4\mathcal{S}(q)}{(\mathbf{p}-\mathbf{q})^2 q^8} = (3.6) = \\
&= \frac{1}{2}\int_0^\infty \frac{dq}{q^4}\mathcal{S}\left(\sqrt{q}\right)\min\{p^4,q^2\}\left(\max\{p^2,q\}-\frac{1}{3}\min\{p^2,q\}\right) \\
&\sim \frac{1}{2}\left(\ln^2 p - \ln p + \frac{21+2\pi^2}{24}\right)p^4.
\end{aligned}
$$

since it is convergent, it is enough to find leading asymptotic, however, it happens to be not enough to find number under the logarithm. What was left is the following integral

$$
\mathcal{S}^{(2a)}(p) - \tilde{\mathcal{S}}^{(2a)}(p) = -\left(\frac{16\pi}{3}\right)\int \frac{(d\mathbf{q})[\mathbf{p}\times\mathbf{q}]^4\mathcal{S}(q)}{\left((\mathbf{p}-\mathbf{q})^2+1\right)q^8} \sim \frac{-1}{2}p^4\int_0^\infty \frac{dq}{q^2}\frac{\mathcal{S}\left(\sqrt{q}\right)}{1+q} = \left(3-\pi^2\right)\frac{p^4}{12}.
$$

As a result,

$$
\frac{T\Sigma^{(2a)}_{\mathbf{p}}}{\varkappa p^4} \sim \frac{2}{d_c^2}\left(\ln^2\left[\sqrt{2}\frac{p}{p_*}\right] - \ln\left[\sqrt{2}\frac{p}{p_*}\right] + \frac{33-2\pi^2}{24}\right).
$$

Upset diagram Expression for such diagram is

$$
\begin{aligned}
T\Sigma^{(2b)}_{\mathbf{p}} &= d_c\int (d\mathbf{q}d\mathbf{k})\left(\frac{Y_{\mathbf{q}}}{T}\right)^2 G_{\mathbf{p}-\mathbf{q}}G_{\mathbf{k}-\mathbf{q}}G^2_{\mathbf{k}}T\Sigma^{(1)}_{\mathbf{k}}[\mathbf{p}\times\hat{\mathbf{q}}]^4[\mathbf{k}\times\hat{\mathbf{q}}]^4 \\
&= -\frac{2}{d_c^2}\varkappa p_*^4\mathcal{S}^{(2b)}\left(\sqrt{2}\frac{p}{p_*}\right), \\
\mathcal{S}^{(2b)}(p) &= \left(\frac{16\pi}{3}\right)^2\int \frac{(d\mathbf{q})[\mathbf{p}\times\mathbf{q}]^4}{(q^4+q^2)(\mathbf{p}-\mathbf{q})^4}\int \frac{(d\mathbf{k})}{k^8}\frac{[\mathbf{k}\times\mathbf{q}]^4}{(\mathbf{k}-\mathbf{q})^4}\mathcal{S}(k) = (3.5) = \\
&= \frac{1}{4}\int_0^\infty \frac{\min\{q^2,p^4\}}{(q^2+q)^2}\int_0^\infty \frac{dk}{k^4}\min\{k^2,q^2\}\mathcal{S}\left(\sqrt{k}\right) \\
&\sim \frac{p^4}{2}\left(\ln^2 p - \ln p + \frac{3}{8} - \frac{\pi^2}{12}\right).
\end{aligned}
$$

As a result,

$$
\frac{T\Sigma^{(2b)}_{\mathbf{p}}}{\varkappa p^4} \sim \frac{4}{d_c^2}\left(-\ln^2\left[\sqrt{2}\frac{p}{p_*}\right] + \ln\left[\sqrt{2}\frac{p}{p_*}\right] - \frac{3}{8} + \frac{\pi^2}{12}\right).
$$

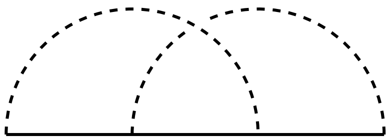
Sum of SCSA–like diagrams Two SCSA–like second order corrections together give

$$
d_c^2\frac{T\Sigma^{(2ab)}_{\mathbf{p}}}{\varkappa p^4} = -2\ln^2\left[\sqrt{2}\frac{p}{p_*}\right] + 2\ln\left[\sqrt{2}\frac{p}{p_*}\right] + \frac{15+2\pi^2}{12}.
$$

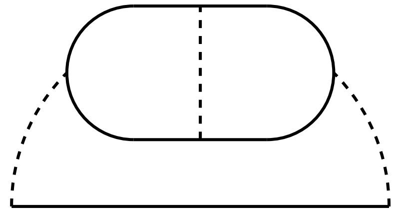
So good so far. Next, I'm interested in Non–SCSA corrections.

3.3.3 Second order. Non–SCSA contributions

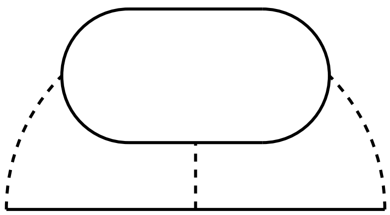
In order find $\delta\eta_2$ I need to calculate following four diagrams.



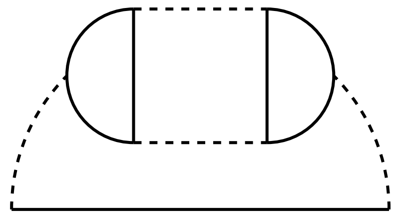
(12c) Crossed diagram



(12d) Pill diagram



(12e) Omnidroid diagram



(12f) Double–pill diagram

Figure 12: Non–SCSA contributions

Here I provide the summary of the results and details of calculation are presented in the next paragraphs. Asymptotics of the diagrams on Figure 12 at $p \ll p_*$ are as follows.

$$d_c^2 \frac{T\Sigma_{\mathbf{p}}^{(2c)}}{\varkappa p^4} = -\frac{7}{3} \ln \left[\frac{p}{p_*}\right] + \text{const}, \tag{3.10}$$

$$d_c^2 \frac{T\Sigma_{\mathbf{p}}^{(2d)}}{\varkappa p^4} = +2 \ln \left[\frac{p}{p_*}\right] + \text{const}, \tag{3.11}$$

$$d_c^2 \frac{T\Sigma_{\mathbf{p}}^{(2e)}}{\varkappa p^4} = +\frac{58}{27} \ln \left[\frac{p}{p_*}\right] + \text{const}, \tag{3.12}$$

$$d_c^2 \frac{T\Sigma_{\mathbf{p}}^{(2f)}}{\varkappa p^4} = -\frac{3+68\zeta(3)}{27} \ln \left[\frac{p}{p_*}\right] + \text{const}. \tag{3.13}$$

Thus, according to (3.9), difference between SCSA and true value of critical exponent is seen in $1/d_c^2$–order and equals to

$$\delta\eta_2 = \lim_{d_c\to\infty} \frac{\eta - \eta_{\text{SCSA}}}{d_c^{-2}} = -\frac{7}{3} + 2 + \frac{58}{27} - \frac{3+68\zeta(3)}{27} = \frac{46-68\zeta(3)}{27} \simeq -1.32.$$

Critical exponent from Hooke's law then $\alpha = \alpha_{\text{SCSA}} + \delta\alpha_2/d_c^2$ with $\delta\alpha_2 = 2(\alpha_{\text{SCSA}}/\eta_{\text{SCSA}})^2\delta\eta_2$.

$$\frac{1}{1+2\alpha} = \frac{1}{1+2\alpha_{\text{SCSA}}}\left(1 - \frac{33-2\sqrt{15}}{21}\frac{\delta\eta_2}{d_c^2} + \dots\right) \simeq \frac{1}{1+2\alpha_{\text{SCSA}}}\left(1 + \frac{1.6}{d_c^2} + \dots\right).$$

3.4 Absolute Poisson ratio

I would like to find absolute Poisson ratio in the regime $\sigma \ll \sigma_*$ of small stress.

$$\nu = \frac{\nu_0 - Y_0 \delta K_2/\sigma}{1 + Y_0 \delta K_1/\sigma} \approx -\frac{\delta K_2}{\delta K_1}.$$

I would like to construct expansion in $1/d_c$ using perturbation theory in screened interaction developed in the previous section. However, I notice that zeroth order phonon correlator diverges at small momenta

$$\delta K_\beta^{(0)} = K_\beta^{(0)}(0) - K_\beta^{(0)}(\sigma) = d_c T \int \frac{(d\mathbf{p}) p_{(\beta)}^2 \sigma p_1^2}{\varkappa p^4 (\varkappa p^4 + \sigma p_1^2)}.$$

To see that explicitly I rewrite integrals as follows.

$$\begin{aligned} \delta K_1^{(0)} - \delta K_2^{(0)} &= \frac{\ln 2}{4\pi} \frac{d_c T}{\varkappa}, \\ \delta K_1^{(0)} + \delta K_2^{(0)} &= 2 \left[\int \frac{\frac{\sigma}{\varkappa}(d\mathbf{p})}{p^2 (p^2 + \frac{\sigma}{\varkappa})} - \frac{\ln 2}{4\pi} \right] \frac{d_c T}{\varkappa}. \end{aligned}$$

Exact correlator does not have such problem since self–energy removes the divergence.

$$\delta K_\beta = d_c T \int \left[\frac{1}{\varkappa p^4 - T\Sigma_p^{\sigma=0}} - \frac{1}{\varkappa p^4 + \sigma p_1^2 - T\Sigma_{\mathbf{p}}} \right] p_\beta^2 (d\mathbf{p})$$

Scaling Ansatz (SA) tells me that in the absence of stress $\sigma = 0$ renormalized bending rigidity behaves as

$$\varkappa_{\mathbf{p}} = \varkappa - p^{-4} T \Sigma_p^{\sigma=0} \sim \varkappa \left(A \frac{p_*}{p} \right)^\eta, \qquad p^2 \ll p_*^2 = \frac{3}{16\pi} \frac{d_c Y_0 T}{\varkappa^2},$$

where critical exponent η is believed to be close to 0.8 and number A is of order unity, which will be hidden in the redefinition of p_* till the end of that section.

It is clear that in the case of finite stress such behavior is only valid when characteristic momenta are larger enough to disregard σp^2 term, i.e. for $p \gg p_\sigma = p_0 p_*^{-\alpha}$, $\alpha = \eta/(2-\eta)$. There are some assumptions about how rigidity renormalizes at finite strain [16], however knowing asymptotic behavior of $\varkappa_{\mathbf{p}}$ at $p \ll p_\sigma$ is not enough to calculate Poisson ratio since integral for correlation function δK comes from all momenta $p \ll p_*$ as it may be seen from the following model.

3.4.1 Oversimplified model

Here I calculate correctional function in the framework of the model

$$\varkappa_{\mathbf{p}}^{\sigma}=\varkappa\begin{cases}1, & p_\sigma<p_*<p\\ (p_*/p)^\eta\,, & p_\sigma<p<p_*\\ (p_*/p_\sigma)^\eta\,, & p<p_\sigma<p_*\end{cases}\qquad p_\sigma=xp_0\left(\frac{p_0}{p_*}\right)^{\alpha}=xp_*\left(\frac{p_0}{p_*}\right)^{1+\alpha}\ll p_*$$

I keep number $x\sim 1$ in order to see what part of the answer is going to be cut-off-dependent. In what follows p_* may be considered ultraviolet cut-off since contributions coming from higher momenta have weaker scaling with stress. For $\sigma\ll\sigma_*$

$$\begin{aligned}\frac{\varkappa}{d_cT}\delta K_\beta&\sim\int\frac{\varkappa\sigma p_\beta^2p_1^2(d\mathbf{p})}{\varkappa_pp^4(\varkappa_pp^4+\sigma p_1^2)}+\int_{p<p_\sigma}(d\mathbf{p})\left[\frac{\varkappa p_\beta^2}{\varkappa_{\mathbf{p}}^{\sigma}p^4+\sigma p_1^2}-\frac{\varkappa p_\beta^2}{\varkappa_{\mathbf{p}}p^4+\sigma p_1^2}\right]\\&=\underbrace{\frac{1+\alpha}{4\sin\pi\alpha}\left\langle\hat p_1^{2\alpha}\hat p_\beta^2\right\rangle_{\hat{\mathbf{p}}}}_{A_\beta^{[-1]}\sim d_c}\left(\frac{\sigma}{\sigma_*}\right)^{\alpha}+\underbrace{\int_{p<x}(d\mathbf{p})\left[\frac{p_\beta^2}{p^4+p_1^2}-\frac{p_\beta^2}{p^{4-\eta}+p_1^2}\right]}_{A_\beta^{[1]}\sim d_c^{-1}}\left(\frac{\sigma}{\sigma_*}\right)^{\alpha}\end{aligned}$$

In the limit $d_c\to\infty$ first term behaves as $\alpha^{-1}\sim d_c$ and second turns into zero together with $\eta\propto d_c^{-1}$. One could also see that dependence on cut-off x is only present in the second contribution. In that model expansion of PR in inverse powers of d_c looks like

$$\begin{aligned}\nu&=-\frac{A_2^{[-1]}+A_2^{[1]}+\dots}{A_1^{[-1]}+A_1^{[1]}+\dots}\sim-\frac{\langle\hat p_1^{2\alpha}\hat p_2^2\rangle_{\hat{\mathbf{p}}}}{\langle\hat p_1^{2\alpha+2}\rangle_{\hat{\mathbf{p}}}}\left(1+\frac{A_2^{[1]}}{A_2^{[-1]}}-\frac{A_1^{[1]}}{A_1^{[-1]}}\right)_{\eta\to0}\\&=\frac{-1}{1+2\alpha}\left(1-\eta^2c_x\right)+\mathcal{O}\!\left(\eta^3\right)\end{aligned}$$

Integrals $A_\beta^{[1]}$ strongly depend on cut-off x.

$$\begin{aligned}c_x&=\frac{4\pi}{\eta}\left(A_1^{[1]}-A_2^{[1]}\right)_{\eta\to0}\sim4\pi\int_{p<x}\frac{\hat p_2^2-\hat p_1^2}{(p^2+\hat p_1^2)^2}p^2\ln p\,(d\mathbf{p})=-\int_0^x\frac{\ln p\,dp}{(1+p^2)^{3/2}}\\&=\operatorname{arcsh}x-\frac{x\ln x}{\sqrt{1+x^2}}\end{aligned}$$

Number c_x varies between 0 and $c_1=\operatorname{arcsh}1\simeq0.88$, as $x\to+\infty$ it tends to $\ln2\simeq.69$.

Of course, such oversimplified model could not produce correct numbers since integrals are determined by full crossover area $p\sim p_\sigma$ and exact $\varkappa_{\mathbf{p}}^{\sigma}$ becomes strongly anisotropic at $p\ll p_\sigma$. Yet that model happens to correctly predict shown below fact that naive answer $(1+2\alpha)^{-1}$ is valid up to the first two orders in $1/d_c$.

3.4.2 Regular perturbation theory

In order to cope with divergence, yet be able to develop perturbation theory in $1/d_c$, I propose to expand δK_β in $\delta\Sigma_{\mathbf{p}} = \Sigma_{\mathbf{p}}^{\sigma=0} - \Sigma_{\mathbf{p}}$.

$$\begin{aligned} \delta K_\beta &= d_c T \int \left[\frac{1}{\varkappa p^4 - T\Sigma_p^{\sigma=0}} - \frac{1}{\varkappa p^4 + \sigma p_1^2 - T\Sigma_p^{\sigma=0}} + \frac{\delta\Sigma_{\mathbf{p}}}{(\varkappa p^4 + \sigma p_1^2 - T\Sigma_p^{\sigma=0})^2} + \dots \right] p_\beta^2 (d\mathbf{p}) \\ &= \underbrace{d_c T \int \frac{\sigma p_\beta^2 p_1^2 (d\mathbf{p})}{\varkappa_p p^4 (\varkappa_p p^4 + \sigma p_1^2)}}_{\delta K_\beta^{[0]} \sim d_c^2} + \underbrace{d_c T \int \frac{p_\beta^2 (d\mathbf{p})}{(\varkappa_p p^4 + \sigma p_1^2)^2} T\delta\Sigma_{\mathbf{p}}^{(1)}}_{\delta K_\beta^{[1]} \sim d_c^0} + \underbrace{d_c T \int (d\mathbf{p}) p_\beta^2 \dots}_{\delta K_\beta^{[2]} \sim d_c^{-1}} \end{aligned} \tag{3.14}$$

Here I note that in the last line I defined $\delta K_\beta^{[1]}$ with $\delta\Sigma_{\mathbf{p}} = \delta\Sigma_{\mathbf{p}}^{(1)}$. Hence $\delta K_\beta^{[2]}$ contains contributions both like $[\delta\Sigma_{\mathbf{p}}^{(1)}]^2$ and like $\delta\Sigma_{\mathbf{p}}^{(2)}$.

Since $\Sigma_{\mathbf{p}}$ has clear expansion in $1/d_c$, expression (3.14) allows me to find δK_β up to desired order in $1/d_c$. Problem is that I actually have two small parameters now and strictly speaking I need to take limit $\sigma/\sigma_* \to 0$ first, i.e. I have to sum up logarithms $\ln \sigma/\sigma_*$ in all orders in $1/d_c$. Let me forget about that for a moment and take a closer look at $\delta K_\beta^{[0]}$.

Leading term is convergent and may be found in the limit $\sigma \ll \sigma_*$ in terms of η.

$$\begin{aligned} \delta K_\beta^{[0]} &\sim \frac{d_c T}{2\pi} \int_0^{p_*} \frac{dp}{p} \left\langle \frac{\sigma \hat{p}_\beta^2 \hat{p}_1^2}{\varkappa (p_*/p)^\eta (\varkappa p_*^\eta p^{2-\eta} + \sigma \hat{p}_1^2)} \right\rangle_{\hat{\mathbf{p}}} \\ &\sim \frac{d_c T}{2\pi} \int_0^\infty \frac{dp \left\langle \hat{p}_1^{2\alpha} \hat{p}_\beta^2 \right\rangle_{\hat{\mathbf{p}}}}{p^{1-\eta}(p^{2-\eta}+1)} \frac{\sigma^\alpha}{(\varkappa p_*^\eta)^{\alpha+1}}, \qquad \alpha = \frac{\eta}{2-\eta}, \quad \eta = \frac{2\alpha}{1+\alpha}, \\ &= \frac{d_c T}{\varkappa} \left(\frac{\sigma}{\sigma_*} \right)^\alpha I_\alpha \left\langle \hat{p}_1^{2\alpha} \hat{p}_\beta^2 \right\rangle_{\hat{\mathbf{p}}}, \qquad I_\alpha = \int_0^\infty \frac{p^{\eta-1} dp/2\pi}{(p^{2-\eta}+1)} = \frac{1+\alpha}{4\sin \pi\alpha}. \end{aligned}$$

Thrown away part of integral is of order $\sigma/\sigma_* \ll (\sigma/\sigma_*)^\alpha$.

$$\frac{Y_0 \delta K_\beta^{[0]}}{\sigma} \sim \frac{16\pi}{3} I_\alpha \left\langle \hat{p}_1^{2\alpha} \hat{p}_\beta^2 \right\rangle_{\hat{\mathbf{p}}} \left(\frac{\sigma}{\sigma_*} \right)^{\alpha-1}, \qquad \sigma \ll \sigma_*.$$

Angle averages may be expressed via Euler Γ–function, but for the PR only their ratio is important.

$$\nu^{[0]} = -\frac{\delta K_2^{[0]}}{\delta K_1^{[0]}} = -\frac{\left\langle \hat{p}_1^{2\alpha} \hat{p}_2^2 \right\rangle_{\hat{\mathbf{p}}}}{\left\langle \hat{p}_1^{2\alpha+2} \right\rangle_{\hat{\mathbf{p}}}} = -\frac{1}{1+2\alpha} = -1 + \frac{2}{d_c} + \mathcal{O}\left(\frac{1}{d_c^2} \right).$$

Since (3.14) is expansion in $1/d_c$, it only makes sense to keep α up to the desired order in $1/d_c$.

Hypothesis I see that both logarithmic series next to d_c^2 and d_c summed up to the same function of σ. I make a hypothesis that in the limit $\sigma \ll \sigma_*$ dependence of δK_β on σ is same as for $\delta K_\beta^{[0]}$ and only number in front is renormalized.

$$\frac{\varkappa}{T}\delta K_\beta = C_\beta(d_c)\left(\frac{\sigma}{\sigma_*}\right)^\alpha, \qquad C_\beta(d_c) = d_c I_\alpha \left\langle \hat{p}_1^{2\alpha}\hat{p}_\beta^2\right\rangle_{\hat{\mathbf{p}}}\left(1+\frac{\#_\beta^{(0,0)}}{d_c^2}+\mathcal{O}\left(\frac{1}{d_c^3}\right)\right).$$

That assumption may be checked in every order of perturbation theory, where I can neglect difference between $\Sigma_{\mathbf{p}}$ and $\Sigma_{\mathbf{p}}^{\sigma=0}$ in every Green's function since I already have $\delta\Sigma_{\mathbf{p}}$ in front. Let me show how it is done for $\delta K^{[1]}$.

$$\Pi_{\mathbf{q}}^{(\eta)} = \frac{d_c}{T}\int \frac{T^2[\mathbf{k}\times\hat{\mathbf{q}}]^4(d\mathbf{k})}{(\varkappa p_*^\eta k^{4-\eta}+\sigma k_1^2)(\varkappa p_*^\eta(\mathbf{q}-\mathbf{k})^{4-\eta}+\sigma(q_1-k_1)^2)} = \frac{d_c T}{\varkappa^2 p_\sigma^2}\left(\frac{p_\sigma}{p_*}\right)^{2\eta} P_\eta\left(\frac{\mathbf{q}}{p_\sigma}\right),$$

$$T\Sigma_{\mathbf{p}}^{(\eta)} = \int\left[\frac{2T}{\Pi_{\mathbf{q}}}\frac{[\mathbf{p}\times\hat{\mathbf{q}}]^4(d\mathbf{q})}{\varkappa p_*^\eta(\mathbf{p}-\mathbf{q})^{4-\eta}+\sigma(p_1-q_1)^2} - \frac{2T}{\Pi_{\mathbf{q}}^{\sigma=0}}\frac{[\mathbf{p}\times\hat{\mathbf{q}}]^4(d\mathbf{q})}{\varkappa p_*^\eta(\mathbf{p}-\mathbf{q})^{4-\eta}}\right] = \frac{\varkappa}{d_c}p_\sigma^4\left(\frac{p_*}{p_\sigma}\right)^\eta S_\eta\left(\frac{\mathbf{p}}{p_\sigma}\right),$$

where Π_η and S_η are some dimensionless functions.

$$\delta K_\beta^{[1]} = \frac{d_c T}{\varkappa}\left(\frac{p_\sigma}{p_*}\right)^{2\eta}\int\frac{(d\mathbf{p})p_\beta^2 S_\eta(\mathbf{p})}{(p^{4-\eta}+p_1^2)^2} = \#_\beta(\alpha)\frac{d_c T}{\varkappa}\left(\frac{\sigma}{\sigma_*}\right)^\alpha.$$

where characteristic scale is

$$p_\sigma = p_0\left(\frac{p_0}{p_*}\right)^\alpha = p_*\left(\frac{p_0}{p_*}\right)^{1+\alpha} \ll p_0 \ll p_*$$

It follows from that hypothesis is that information about Poisson ratio is encoded in numbers $\#_\beta^{(n,0)}$, in particular, number $\#^{(0,0)}$ may be found from $\delta K^{[1]}$ with $\varkappa_{\mathbf{p}} = \varkappa$.

Correction to leading order value $\nu^{[0]}$ of Poisson ratio comes from $\delta K_\beta^{[1]}$. In the limit $d_c \to \infty$ correlator $\delta K_\beta^{[1]}$ is simply a number

$$\delta K_\beta^{[1]} \underset{d_c\to\infty}{=} d_c T\int\frac{p_\beta^2(d\mathbf{p})}{(\varkappa p^4+\sigma p_1^2)^2}\int\left[\frac{2T}{\Pi_{\mathbf{q}}}\frac{[\mathbf{p}\times\hat{\mathbf{q}}]^4}{\varkappa(\mathbf{p}-\mathbf{q})^4+\sigma(p_1-q_1)^2} - \frac{2T}{\Pi_{\mathbf{q}}^{\sigma=0}}\frac{[\mathbf{p}\times\hat{\mathbf{q}}]^4}{\varkappa(\mathbf{p}-\mathbf{q})^4}\right](d\mathbf{q})$$
$$= \frac{2T}{\varkappa}\int\frac{p_\beta^2(d\mathbf{p})}{(p^4+p_1^2)^2}\int\left[\frac{1}{P(\mathbf{q})}\frac{[\mathbf{p}\times\hat{\mathbf{q}}]^4}{(\mathbf{p}-\mathbf{q})^4+(p_1-q_1)^2} - \frac{1}{P_0(\mathbf{q})}\frac{[\mathbf{p}\times\hat{\mathbf{q}}]^4}{(\mathbf{p}-\mathbf{q})^4}\right](d\mathbf{q})$$

where dimensionless polarization operators are defined according to

$$\Pi^{\sigma=0}_{\mathbf{q}} = \frac{d_c T}{\varkappa^2}\int \frac{[\mathbf{k}\times\hat{\mathbf{q}}]^4 (d\mathbf{k})}{k^4(\mathbf{q}-\mathbf{k})^4} = \frac{d_c T}{\varkappa\sigma} P_0\left(\sqrt{\frac{\varkappa}{\sigma}}\mathbf{q}\right) = \frac{3}{16\pi}\frac{d_c T}{\varkappa^2}\frac{1}{q^2},$$
$$\Pi_{\mathbf{q}} = \frac{d_c}{T}\int \frac{T^2[\mathbf{k}\times\hat{\mathbf{q}}]^4 (d\mathbf{k})}{(\varkappa k^4 + \sigma k_1^2)(\varkappa(\mathbf{q}-\mathbf{k})^4 + \sigma(q_1-k_1)^2)} = \frac{d_c T}{\varkappa\sigma} P_1\left(\sqrt{\frac{\varkappa}{\sigma}}\mathbf{q}\right).$$

Dimensionless polarization operator for uniaxial stress $P_1(\mathbf{q})$ is calculated in the appendix.

Answer Since $\delta K^{[0]}_\beta \sim \frac{d_c^2}{8\pi}$, $d_c \to \infty$, Poisson ration in the first non–trivial order may be written as

$$\begin{aligned}\nu = -\frac{\delta K^{[0]}_2 + \delta K^{[1]}_2 + \ldots}{\delta K^{[0]}_2 + \delta K^{[1]}_2 + \ldots} &= -\frac{\delta K^{[0]}_2}{\delta K^{[0]}_1}\left[1 + \frac{\delta K^{[1]}_2}{\delta K^{[0]}_2} - \frac{\delta K^{[1]}_1}{\delta K^{[0]}_1} + \ldots\right] \\ &= -\frac{\delta K^{[0]}_2}{\delta K^{[0]}_1}\left[1 + \frac{8\pi}{d_c^2}\frac{\varkappa}{T}\left(\delta K^{[1]}_2 - \delta K^{[1]}_1\right) + \ldots\right] \\ &= -\frac{1}{1+2\alpha}\left[1 + \frac{c}{d_c^2} + \mathcal{O}\left(\frac{1}{d_c^3}\right)\right]\end{aligned}$$

where number $c = 8\pi(\#^{(0,0)}_2 - \#^{(0,0)}_1)$ is of interest.

$$\frac{c}{16\pi} = \int_{\mathbf{q},\mathbf{k}} \left[\frac{(k_2^2 - k_1^2)[\mathbf{k}\times\hat{\mathbf{q}}]^4}{P_1(\mathbf{q})(k^4 + k_1^2)^2((\mathbf{k}-\mathbf{q})^4 + (k_1-q_1)^2)} - \frac{(k_2^2 - k_1^2)[\mathbf{k}\times\hat{\mathbf{q}}]^4}{P_0(\mathbf{q})(k^4 + k_1^2)^2(\mathbf{k}-\mathbf{q})^4}\right]$$

Such integral may be computed numerically with $P_0(\mathbf{q}) = \frac{3}{16\pi}q^{-2}$ and expression for $P_1(\mathbf{q})$ is provided in the appendix. The answer is $c = -0.56 \pm 0.01$, which allows me conclude that true expression for absolute Poisson ratio in non–linear regime $\sigma \ll \sigma_*$ is not given by simple expression as $1/(1+2\alpha)$, but a more complex function.

In the next section I relate critical exponent α to its approximate (self–consistent) value $\alpha_{\rm sc}$ in order to find Poisson ratio up to $1/d_c^2$.

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Chapter 4

Numerical simulations

4.1 Overview

In this section I sum up and give brief comments on recent published numerical studies of universal regime of the model (2.2). Papers from previous century are added selectively, for more compete list reader is directed to recent review [5, Fig. 1]. Some notions in this table

Year	Author	Method	p_8	Size, L^2	η	$\nu_{\sigma=0}$
1996	[17] Bowick	Hausdorf dim Space correlator	$\approx .4$ $\approx .4$	128^2	$.72 \pm .12$ $\approx .6$	
1997	[18] Falcioni *et al*	Space correlator	$\approx .4$	128^2	$\approx .62$	$-.3 \pm 0.06$
2001	[4] Bowick *et al*	Self–avoiding		95^2		$-.37 \pm 0.6$
2009	[19] Los *et al*	Molecular dynamics	–	190^2	$\approx .85$	
2012	[20] Troster	Fourier correlator Fourier Green	.44	$640^2(40^2)$	$.795 \pm .01$ $\approx .761$	
2016	[21] Los *et al*	Space correlator	.4	195^2	≈ 0.84	$+.275$
2019	[#] Saykin	Fourier Green	.3	360^2	$.78 \pm .02$	$-.76 \pm .05$

Table 4.1: Selected reports on numerical values of critical indices.

require additional clarification. Most of the papers use Monte Carlo simulations of action (2.2), some work in $\mathbf{r}$–space, some in Fourier $\mathbf{k}$–space, some papers study behavior of Green function itself, some study correlator

$$K_{\alpha\beta} = \int d\mathbf{x}(\partial_\alpha \mathbf{r} \cdot \partial_\beta \mathbf{r})$$

instead. Column p_8 contains information about interaction force. It is useful to measure it in dimensionless number

$$p_8 = p_* a = a\sqrt{\frac{3}{16\pi}\frac{Y_0 T}{\varkappa^2}},$$

where I use $a = 2.46$ Å for graphene lattice constant, $Y_0 \simeq 22$ eV·Å^{-2} and $\varkappa \simeq 1.1$ eV. For $T = 300$ K it gives $p_8 = .41$.

One can see that there are many discrepancies between reported data which calls for explanation. Below I reproduce method used in [20] and calculate η and Poisson ratio $\nu_{\sigma=0}$.

4.2 Simulation description

I use Metropolis–Hastings algorithm and run Monte–Carlo simulations with weight function $e^{-F/T}$ and I use (3.1) expression with $\sigma = 0$, which in dimensionless form looks as follows.

$$\mathcal{F} = \sum_{\mathbf{q}\neq 0} \left\{ \frac{1}{2} \left(\frac{2\pi q}{L} \right)^4 \left| h_{[\mathbf{q}]} \right|^2 + \frac{2\pi}{3} \frac{p_8^2}{L^2} \left| S_{[\mathbf{q}]} \right|^2 \right\}$$

with the following function

$$S_{[\mathbf{q}]} = \sum_{\mathbf{k}\neq 0} \left[\frac{2\pi}{L} \mathbf{k} \times \hat{\mathbf{q}} \right]^2 \left(h_{[\mathbf{k}]} \cdot h_{[\mathbf{k}+\mathbf{q}]} \right)$$

and all momenta replaced by sin–value

$$\frac{2\pi}{L} k_1 \mapsto \sin\left(\frac{2\pi}{L} k_1 \right), \qquad q^2 \mapsto 4\sin^2\left(\frac{\pi}{L} q_1 \right) + 4\sin^2\left(\frac{\pi}{L} q_2 \right).$$

Vector product is defined by its value in first Brillouin zone and continued from there by periodicity.

Running $\sim 10^5$ Monte–Carlo steps I find Green function

$$\mathcal{G}_{\mathbf{p}} = \left\langle \left| h_{[\mathbf{p}]} \right|^2 \right\rangle_{\text{MC}} = \frac{1}{\varkappa_p} \left(\frac{2\pi p}{L} \right)^{-4}.$$

I choose phase space exploration step so that acceptance rate varies between 35% and 55%. As it is mentioned in [20] step must be momenta–dependent and could be heuristically choose from relation $d_{[\mathbf{k}]} \sim h_{[\mathbf{k}]}$.

From $\mathcal{G}_{\mathbf{k}}$ dependence I extract value of critical exponent η. I present results on Figure 13 and Figure 14. As it could be seen the smallest wave vectors behave themselves inconsistently, which my be called finite–size effect [20]. That's why I only use third and higher momenta for fit. I report value

$$\eta = 0.78 \pm 0.02.$$

That value is in good agreement with numerical paper [20] and «second–order» self–consistent approximation [6].

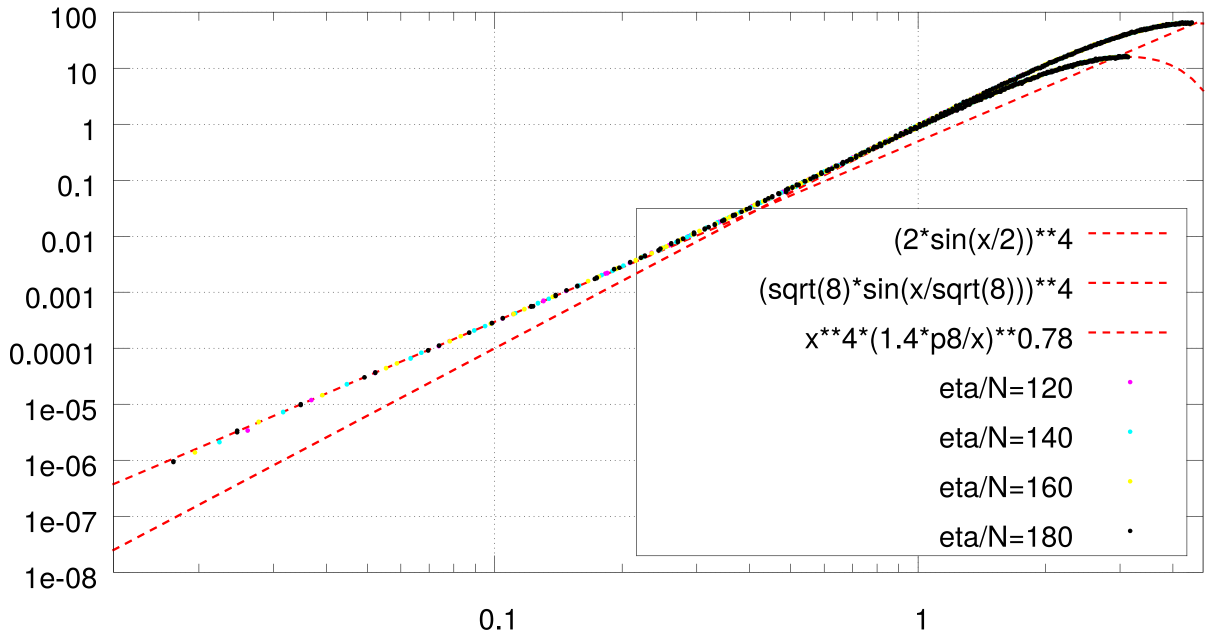

Figure 13: Log–log plot of $\mathcal{G}_{\mathbf{q}}^{-1}$ for different lattice sizes $L = 200$, 240, 280, 320, 360.

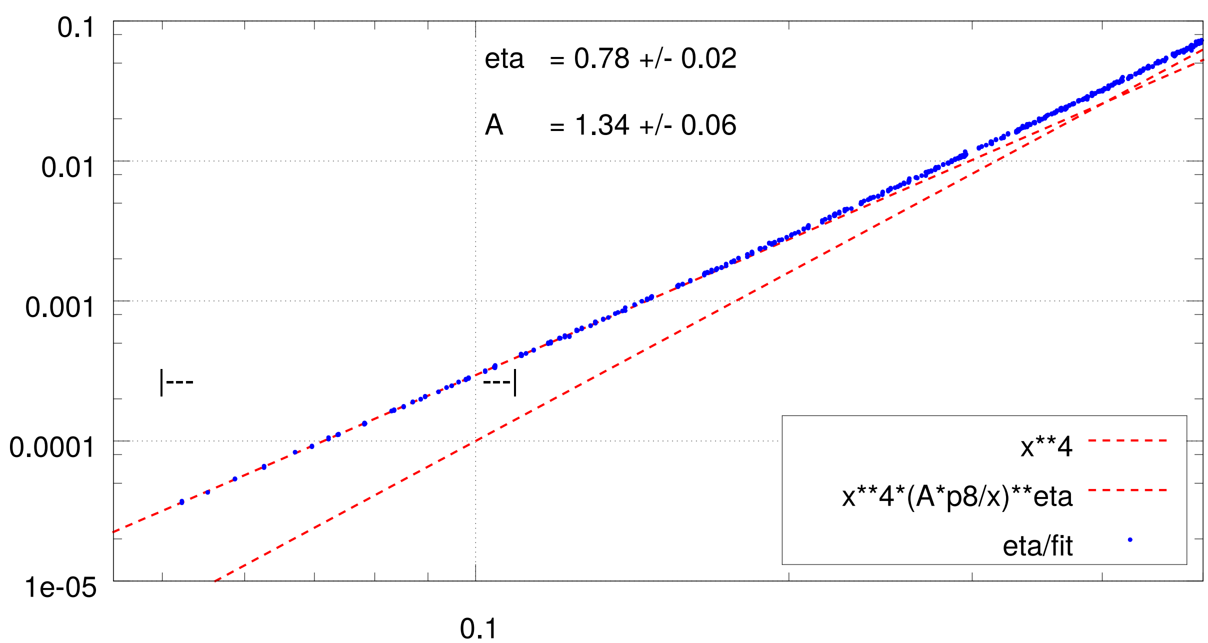

Figure 14: Close–up view. Fit region is marked. Presented lattice sizes $L = 200$, 220, 240, 260, 280, 300, 320, 340, 360.

Poisson ratio I also calculate value of Poisson ratio in non–linear regime as

$$\nu_{\sigma=0} = -\frac{\sum_{|\mathbf{q}|<q_8} \left\langle K^x_{[\mathbf{q}]} K^y_{[\mathbf{q}]} \right\rangle - \left\langle K^x_{[\mathbf{q}]} \right\rangle \left\langle K^y_{[\mathbf{q}]} \right\rangle}{\sum_{|\mathbf{q}|<q_8} \left\langle K^x_{[\mathbf{q}]} K^x_{[\mathbf{q}]} \right\rangle - \left\langle K^x_{[\mathbf{q}]} \right\rangle^2}, \qquad K^\alpha_{[\mathbf{q}]} = \sin^2\left(\frac{2\pi}{L} k_\alpha\right) \left|h_{[\mathbf{q}]}\right|^2$$

and report the value

$$\nu_{\sigma=0} = -0.76 \pm 0.05.$$

4.2.1 Algorithm

The idea of effective calculation is expounded in [22]. Key idea is to keep $S_{[\mathbf{q}]}$ and $dS_{[\mathbf{q}]}$ arrays and evaluate only $dS_{[\mathbf{q}]}$ each Monte–Carlo step. Here is the listing of the code that does such evaluation.

```
for (q1 = 0; q1 < L; q1++)
  for (q2 = 0; q2 < L; q2++) {
    if (!q1 && !q2) continue;
    double p = sin[k1]*sin[q2]-sin[k2]*sin[q1]; p *= p;
    double kq = sin[k2+q2]*sin[q1]-sin[k1+q1]*sin[q2]; kq *= kq;
    double qk = sin[k1-q1]*sin[q2]-sin[k2-q2]*sin[q1]; qk *= qk;
    complex s = p*conj(h[k1+q1][k2+q2])*z;
    s += kq*h[-k1-q1][-k2-q2]*z;
    s += qk*h[k1-q1][k2-q2]*conj(z);
    s +=  p*conj(h[q1-k1][q2-k2])*conj(z);
    if (!((q1+2*k1)%L) && !((q2+2*k2)%L))
      s += p*z*z*d;
    if (!((q1-2*k1+2*L)%L) && !((q2-2*k2+2*L)%L))
      s += p*conj(z)*conj(z)*d;
    s *= d/Q[q1][q2];
    dS[q1][q2] = s;
    w += creal((2*S[q1][q2]+s)*conj(s));
}
```

Main body of simulate step function is printed on the next page. Here I note these listings are not one–to–one exact to a working code. Reader may find source code github.com/saykind.

```c
#include <complex.h>
#include <omp.h>

int simulate(complex **h, complex **S, complex **dS, double **g) {
  int L = 2*N+1;
  int k1 = rand()%L-N, k2 = rand()%L-N, q1, q2;
  if (!k1 && !k2) {simulate(h, S, dS, g); return 0;}
  double w = 0, A = Q[k1][k2], d = 2.6/A/pow(1+Y/A,.13); A *= A;

  complex z = (1.*rand()/RAND_MAX-.5) + (1.*rand()/RAND_MAX-.5)*I;
  #pragma omp parallel for collapse(2) reduction(+:w)
  for (q1 = 0; q1 < L; q1++)
    for (q2 = 0; q2 < L; q2++) {
      if (!q1 && !q2) continue;
      /* Calculation of s=dS[q1][q2] */
      dS[q1][q2] = s;
      w += creal((2*S[q1][q2]+s)*conj(s));
  }
  w *= -Y/L/L;
  w -= A*creal((2*h[k1][k2] + d*z)*conj(z))*d;
  if (w > log(1.*rand()/RAND_MAX)) {
    h[k1][k2] += d*z;
    h[(L-k1)%L][(L-k2)%L] += d*conj(z);
    #pragma omp parallel for collapse(2)
    for (q1 = 0; q1 < L; q1++)
      for (q2 = 0; q2 < L; q2++)
        S[q1][q2] += dS[q1][q2];
  }
  if (c && g) {
    double a = creal(h[k1][k2]*conj(h[k1][k2]));
    g[k1][k2] += a;
    g[(L-k1)%L][(L-k2)%L] += a;
  }
  return 0;
}
```

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Chapter 5
Summary

5.1 Results

5.1.1 Critical exponents

I have found critical exponent η and its derivative $\alpha = \frac{\eta}{2-\eta}$ that characterize universal regime (system forgets about material–dependent characteristics μ, λ, $\varkappa$) of large enough two–dimensional crystal under not strong stress ($\sigma_L \ll \sigma \ll \sigma_*$) perturbatively in $1/d_c$ up to the first non–trivial[1] order.

$$\begin{aligned}
\eta &= \frac{2}{d_c} + \frac{73 - 68\zeta(3)}{27}\frac{1}{d_c^2} + \mathcal{O}\big(d_c^{-3}\big)\,, \\
&= \frac{2}{d_c} - \frac{.324}{d_c^2} + \mathcal{O}\big(d_c^{-3}\big)\,. \\
\alpha &= \frac{1}{d_c} + \frac{127 - 68\zeta(3)}{54}\frac{1}{d_c^2} + \mathcal{O}\big(d_c^{-3}\big)\,, \\
&= \frac{1}{d_c} + \frac{.838}{d_c^2} + \mathcal{O}\big(d_c^{-3}\big)\,.
\end{aligned}$$

In particular, my calculations hint that true value of η and α may be considerably different from the values given by self–consistent screening approximation $\eta_{\text{SCSA}} = \frac{4}{1+\sqrt{15}} \approx 0.821$ and $\alpha_{\text{SCSA}} = \frac{1}{7}(1+\sqrt{15}) \approx 0.696$.

$$\begin{aligned}
\eta &= \eta_{\text{SCSA}} - \frac{68\zeta(3) - 46}{108}\eta_{\text{SCSA}}^2 + \mathcal{O}\big(\eta_{\text{SCSA}}^3\big)\,, \\
&= \eta_{\text{SCSA}} - .331\eta_{\text{SCSA}}^2 + \mathcal{O}\big(\eta_{\text{SCSA}}^3\big)\,. \\
\alpha &= \alpha_{\text{SCSA}} - \frac{34\zeta(3) - 23}{27}\alpha_{\text{SCSA}}^2 + \mathcal{O}\big(\alpha_{\text{SCSA}}^3\big)\,, \\
&= \alpha_{\text{SCSA}} - .662\alpha_{\text{SCSA}}^2 + \mathcal{O}\big(\alpha_{\text{SCSA}}^3\big)\,.
\end{aligned}$$

In order to make even very bold estimate of actual values of η an α from these expressions one has to know whether he deals with sign–alternating series or not. I make a wild guess that next term goes with a positive sign, and estimate $\eta \approx .7$ and $\alpha \approx .5$.

[1] Before only the leading term $\eta \sim 2/d_c$ and SCSA approximation were known.

5.1.2 Absolute Poisson ratio

In the fore-mentioned universal regime, Poisson ratio also becomes universal number.

$$\begin{aligned}
-\nu &= 1 - \frac{2}{d_c} + \frac{1.76}{d_c^2} + \mathcal{O}\big(d_c^{-3}\big)\,,\\
&= 1 - \eta + .36\eta^2 + \mathcal{O}\big(\eta^3\big)\,,\\
&= 1 - \eta_{\text{SCSA}} + .69\eta_{\text{SCSA}}^2 + \mathcal{O}\big(\eta_{\text{SCSA}}^3\big)\,,\\
&= 1 - 2\alpha + 3.44\alpha^2 + \mathcal{O}\big(\alpha^3\big)\,,\\
&= 1 - 2\alpha_{\text{SCSA}} + 4.12\alpha_{\text{SCSA}}^2 + \mathcal{O}\big(\alpha_{\text{SCSA}}^3\big)\,.
\end{aligned}$$

I would also like to make a guess about actual value of ν, so I take my chances and suppose that series is alternating in sign, then I would say $\nu \approx -.4$ might be a good guess.

5.1.3 Numerical results

Here I report result of Monte–Carlo simulations.

$$\eta = 0.78 \pm 0.02, \qquad \nu_{\sigma=0} = -0.76 \pm 0.05.$$

Please note that value of Poisson ratio at zero stress does not has much in common with value ν, $\sigma_L \ll \sigma \ll \sigma_*$ discussed in the main paper as it was pointed out in [10].

5.2 Conclusion

Here I give answers to the questions stated at the beginning of the paper.

1. Critical exponent α from anomalous Hooke's law could differ drastically (10%) from SCSA value [9]. There is no theoretical reason to believe it is close to $\alpha_{\text{SCSA}} \approx .7$. Numerical simulations suggest that value is somewhere in the range $\alpha = .64 \pm .2$.

2. The value of Poisson ratio ν in universal regime is an independent critical index. Similarly, there is no reason to believe that either differential or absolute Poisson ratio values are close to their SCSA approximations.

Appendix A

Calculation details

A.1 Second order non–SCSA contributions

Here I give the details of calculation of second–order contributions to the self energy that are present in (3.10, 3.11, 3.12, 3.13).

Second–order. Crossed diagram Here goes diagram depicted at Figure 12–12*c*.

$$T\Sigma^{(2c)}_{\mathbf{p}} = \int \frac{Y_0^2 T^2 (d\mathbf{q}d\mathbf{k})[\mathbf{p}-\mathbf{q}\times\mathbf{k}]^2[\mathbf{p}-\mathbf{k}\times\mathbf{q}]^2[\mathbf{p}\times\mathbf{k}]^2[\mathbf{p}\times\mathbf{q}]^2}{\varkappa^3(q^4+\frac{p_*^2}{2}q^2)(k^4+\frac{p_*^2}{2}k^2)(\mathbf{p}-\mathbf{q})^4(\mathbf{p}-\mathbf{q}-\mathbf{k})^4(\mathbf{p}-\mathbf{k})^4} = \frac{\varkappa p_*^4}{d_c^2}\mathcal{S}^{(2c)}\left(\frac{\sqrt{2}p}{p_*}\right),$$

$$\begin{aligned}\mathcal{S}^{(2c)}(p) &= \left(\frac{16\pi}{3}\right)^2 \int \frac{(d\mathbf{q})[\mathbf{p}\times\mathbf{q}]^2}{(q^4+q^2)(\mathbf{p}-\mathbf{q})^4} \int \frac{[\mathbf{p}\times\mathbf{k}]^2[\mathbf{p}-\mathbf{q}\times\mathbf{k}]^2[\mathbf{p}-\mathbf{k}\times\mathbf{q}]^2}{(k^4+k^2)(\mathbf{p}-\mathbf{q}-\mathbf{k})^4(\mathbf{p}-\mathbf{k})^4}(d\mathbf{k}) \\ &\sim \left(\frac{16\pi}{3}\right)^2 p^4 \int \frac{(d\mathbf{q})q^2[\hat{\mathbf{p}}\times\hat{\mathbf{q}}]^2}{(1+q^{-2})(\mathbf{p}-\mathbf{q})^4} \int \frac{[\hat{\mathbf{p}}\times\hat{\mathbf{k}}]^2[\hat{\mathbf{k}}\times\hat{\mathbf{q}}]^4}{(\mathbf{k}-\mathbf{q})^4}(d\mathbf{k}), \qquad p \ll 1.\end{aligned}$$

I use identity (3.7) and two more expressions

$$\begin{aligned}\left\langle \frac{[\hat{\mathbf{p}}\times\hat{\mathbf{k}}]^2[\hat{\mathbf{k}}\times\hat{\mathbf{q}}]^4}{(\mathbf{k}-\mathbf{q})^4}\right\rangle_{\hat{\mathbf{k}}} &= \frac{3/16}{\max\{k^4,q^4\}} + \left[\frac{4}{\max\{k^4,q^4\}} - \frac{5\min\{k^2,q^2\}}{\max\{k^6,q^6\}}\right]\frac{\cos[2\widehat{\mathbf{p}\mathbf{q}}]}{32}, \\ \left\langle \frac{[\hat{\mathbf{p}}\times\hat{\mathbf{q}}]^2}{(\mathbf{p}-\mathbf{q})^4}\cos[2\widehat{\mathbf{p}\mathbf{q}}]\right\rangle_{\hat{\mathbf{k}}} &= \frac{3\min\{p^2,q^2\}-\max\{p^2,q^2\}}{4\max\{p^4,q^4\}|p^2-q^2|}.\end{aligned}$$

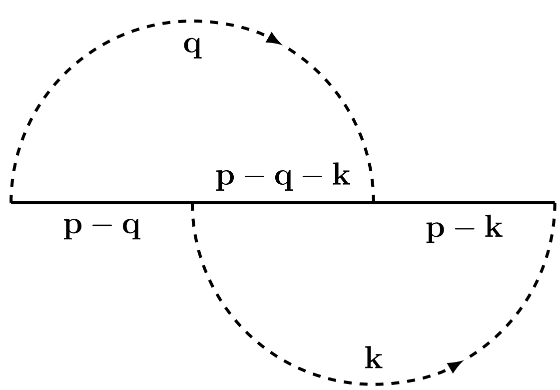

Figure 15: Crossed diagram

Changing the variables $q^2 \mapsto q$, $k^2 \mapsto k$, I come to

$$\begin{aligned}\mathcal{S}^{(2c)}(p) &= \frac{1}{12}\frac{p^4}{4}\int_0^\infty \frac{qdqdk}{1+q^{-1}}\left(\frac{6}{\max\{k^2,q^2\}}\cdot\frac{4}{3}\frac{1}{\max\{p^2,q\}|p^2-q|}+\right.\\ &\left.\quad+\left[\frac{4}{\max\{k^2,q^2\}}-5\frac{\min\{k,q\}}{\max\{k^3,q^3\}}\right]\cdot\frac{2}{3}\frac{3\min\{p^2,q\}-\max\{p^2,q\}}{\max\{p^4,q^2\}|p^2-q|}\right)\\ &= \frac{p^4}{4}\int_0^\infty \frac{dq}{1+q^{-1}}\left(\frac{4/3}{\max\{p^2,q\}|p^2-q|}+\frac{3\min\{p^2,q\}-\max\{p^2,q\}}{6\max\{p^4,q^2\}|p^2-q|}\right)\end{aligned}$$

This integral is not hard to calculate exactly. Finally,

$$\mathcal{S}^{(2c)}(p) \sim -\frac{7}{3}\left(\frac{p}{\sqrt{2}}\right)^4 \ln p + \text{const.}$$

That is the result (3.10).

Second–order. Pill diagram Here is another one.

$$\begin{aligned}T\Sigma^{(2d)}_{\mathbf{p}} &= -\frac{d_c}{2}\frac{Y_0^3T^3}{\varkappa^5}\int\frac{(d\mathbf{q}d\mathbf{r}d\mathbf{k})[\mathbf{p}\times\mathbf{q}]^4[\mathbf{k}\times\mathbf{q}]^2[\mathbf{k}\times\mathbf{r}]^2[\mathbf{k}-\mathbf{r}\times\mathbf{q}]^2[\mathbf{k}-\mathbf{q}\times\mathbf{r}]^2}{(q^4+\frac{p_*^2}{2}q^2)^2(r^4+\frac{p_*^2}{2}r^2)(\mathbf{p}-\mathbf{q})^4k^4(\mathbf{k}-\mathbf{q})^4(\mathbf{k}-\mathbf{r})^4(\mathbf{k}-\mathbf{q}-\mathbf{r})^4}\\ &= \frac{\varkappa p_*^4}{d_c^2}\mathcal{S}^{(2d)}\left(\sqrt{2}\frac{p}{p_*}\right),\end{aligned}$$

$$\mathcal{S}^{(2d)}(p) = -\left(\frac{16\pi}{3}\right)^3\int\frac{[\mathbf{p}\times\mathbf{q}]^4q^2(d\mathbf{q})}{(q^4+q^2)^2(\mathbf{p}-\mathbf{q})^4}\int\frac{(d\mathbf{r}d\mathbf{k})[\mathbf{k}\times\hat{\mathbf{q}}]^2[\mathbf{k}\times\mathbf{r}]^2[\mathbf{k}-\mathbf{r}\times\hat{\mathbf{q}}]^2[\mathbf{k}-\hat{\mathbf{q}}\times\mathbf{r}]^2}{(\cancel{q^2r^4}+r^2)k^4(\mathbf{k}-\hat{\mathbf{q}})^4(\mathbf{k}-\mathbf{r})^4(\mathbf{k}-\hat{\mathbf{q}}-\mathbf{r})^4}$$

$$\mathcal{S}^{(2d)}(p) \sim -\left(\frac{16\pi}{3}\right)^2\frac{f}{2}\int_0^\infty\frac{dqq}{(q^2+q)^2}\min\{p^4,q^2\} \sim \left(\frac{16\pi}{3}\right)^2 fp^4\ln p.$$

So I am interested in calculating number f.

$$f = \int\frac{[\hat{\mathbf{k}}\times\hat{\mathbf{q}}]^2[\hat{\mathbf{k}}\times\hat{\mathbf{r}}]^2[\hat{\mathbf{r}}\times\hat{\mathbf{q}}]^2}{(\mathbf{k}-\hat{\mathbf{q}})^4(\mathbf{k}-\mathbf{r})^2(\mathbf{r}-\hat{\mathbf{q}})^4}[\mathbf{k}-\hat{\mathbf{q}}\times\mathbf{r}-\hat{\mathbf{q}}]^2(d\mathbf{r}d\mathbf{k})$$

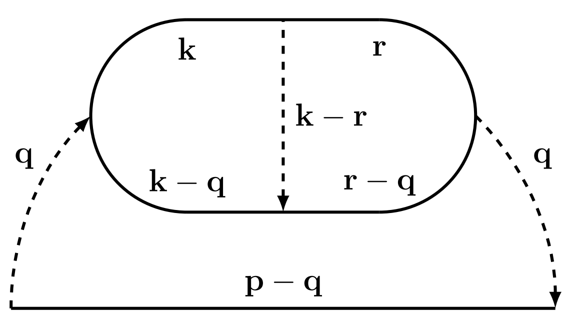

Figure 16: The Eye of Sauron diagram

In order to average over angles I change the variables to $z = \exp\left[i\widehat{\mathbf{kq}}\right]$, $w = \exp\left[i\widehat{\mathbf{rq}}\right]$. I find several potential residues $z = 0,\ k,\ 1/k,\ kw/r,\ rw/k$. Depending on parameters k, r they either lie inside unit circle $|z| = 1$ or not. After evaluating integrals over z I come to expressions with potential residue at $w = 0,\ r,\ 1/r,\ k^2/r,\ r/k^2$. All in all, I have six different cases depending on relation between $k > 1$, $r > 1$, $k > r$. It so happens that relation $k^2 > r$ does not change the answer. Altogether contributions give $f = (1/2)(3/16\pi)^2$. Thus,

$$d_c^2 \frac{T\Sigma_{\mathbf{p}}^{(2d)}}{\varkappa p^4} \sim 2\ln\left[\frac{p}{p_*}\right] + \text{const.}$$

That is the result (3.11). Results (3.12), (3.13) are obtained in similar manner.

A.2 Polarization operator

Here I calculate the polarization operator for the case of uniaxial stress $\sigma_1 = \sigma$, $\sigma_2 = 0$.

$$\Pi_{\mathbf{q}} = \frac{d_c T}{\varkappa\sigma} \int \frac{[\mathbf{k}\times\hat{\mathbf{q}}]^2}{\mathbf{k}^4 + k_1^2} \frac{[(\mathbf{q}-\mathbf{k})\times\hat{\mathbf{q}}]^2}{(\mathbf{q}-\mathbf{k})^4 + (q_1 - k_1)^2} (d\mathbf{k}) = \frac{d_c T}{\varkappa\sigma} P\left(\sqrt{\frac{\varkappa}{\sigma}}\mathbf{q}\right),$$

$$P(\mathbf{q}) = \int (d\mathbf{k}) f_{\mathbf{k}} f_{\mathbf{q}-\mathbf{k}} = \int d^2\mathbf{x}\, e^{-i\mathbf{q}\mathbf{x}} f^2(\mathbf{x}), \qquad f(\mathbf{x}) = \int \frac{[\mathbf{k}\times\hat{\mathbf{q}}]^2}{k^4 + k_1^2} e^{i\mathbf{k}\mathbf{x}} (d\mathbf{k}).$$

Function $f(\mathbf{x})$ may be calculated in coordinates. Let me define

$$f^{(a,b)}(\mathbf{x}) = \int \frac{k_1^a k_2^b}{k^4 + k_1^2} e^{i\mathbf{k}\mathbf{x}} (d\mathbf{k}), \qquad a, b \in \mathbb{N}_0,\ \ a + b \neq 0,$$
$$f^{(0,0)}(\mathbf{x}) = \int \frac{e^{i\mathbf{k}\mathbf{x}} - 1}{k^4 + k_1^2} (d\mathbf{k}),$$

then the function I'm interested in is given by $f(\mathbf{x}) = -[\hat{\mathbf{q}}\times\nabla]^2 f^{(0,0)}(\mathbf{x})$. Unfortunately, I don't know how to calculate $f^{(0,0)}$. Fortunately, $f^{(1,0)}$ and $f^{(2,0)} + f^{(0,2)}$ are possible to calculate.

$$f^{(1,0)}(\mathbf{x}) = \sum_{\pm} \frac{\mp}{2i} \int \frac{e^{i\mathbf{k}\mathbf{x}}(d\mathbf{k})}{k^2 \pm i k_1} = \sum_{\pm} \frac{\mp}{2i} \int \frac{k dk}{2\pi} \frac{J_0(kx)}{k^2 + \frac{1}{4}} e^{\pm x_1/2} = \frac{i}{2\pi} \operatorname{sh}\frac{x_1}{2} K_0 \frac{x}{2}.$$
$$f^{(2,0)}(\mathbf{x}) + f^{(0,2)}(\mathbf{x}) = \sum_{\pm} \frac{1}{2} \int \frac{e^{i\mathbf{k}\mathbf{x}}(d\mathbf{k})}{k^2 \pm i k_1} = \sum_{\pm} \frac{1}{2} \int \frac{k dk}{2\pi} \frac{J_0(kx)}{k^2 + \frac{1}{4}} e^{\pm x_1/2} = \frac{1}{2\pi} \operatorname{ch}\frac{x_1}{2} K_0 \frac{x}{2}.$$

From these expressions, with the help of the property $f^{(a+n,b+m)} = (-i\partial_1)^n(-i\partial_2)^m f^{(a,b)}$ I am able to find following functions.

$$f^{(2,0)} = \frac{1}{4\pi}\left[\operatorname{ch}\frac{x_1}{2} K_0\frac{x}{2} - \frac{x_1}{x}\operatorname{sh}\frac{x_1}{2} K_1 \frac{x}{2}\right],$$
$$f^{(1,1)} = -\frac{1}{4\pi}\frac{x_2}{x}\operatorname{sh}\frac{x_1}{2} K_1\frac{x}{2},$$
$$f^{(0,2)} = \frac{1}{4\pi}\left[\operatorname{ch}\frac{x_1}{2} K_0\frac{x}{2} + \frac{x_1}{x}\operatorname{sh}\frac{x_1}{2} K_1 \frac{x}{2}\right].$$

Now I need to multiple different function in $\mathbf{x}$–space and perform inverse Fourier transform. Let me define $F^{(a,b;c,d)}(\mathbf{x}) = f^{(a,b)}(\mathbf{x}) f^{(c,d)}(\mathbf{x})$ and $F^{(a,b)}(\mathbf{x}) = f^{(a,b)}(\mathbf{x}) f^{(a,b)}(\mathbf{x})$, then polarization operator is given by

$$P(\mathbf{q}) = \hat{q}_1^4 F_{\mathbf{q}}^{(0,2)} + 4\hat{q}_1^2\hat{q}_2^2 F_{\mathbf{q}}^{(1,1)} + \hat{q}_2^4 F_{\mathbf{q}}^{(0,2)} + 2\hat{q}_1^2\hat{q}_2^2 F_{\mathbf{q}}^{(0,2,2,0)} - 4\hat{q}_1^3\hat{q}_2 F_{\mathbf{q}}^{(0,2,1,1)} - 4\hat{q}_1\hat{q}_2^3 F_{\mathbf{q}}^{(2,0,1,1)}. \quad \text{(A.1)}$$

To evaluate these integrals I use the following identities.

$$\int_0^\infty xK_0^2\left(\frac{x}{2}\right)J_0(qx)dx = \frac{2\,\mathrm{arcsh}\,q}{q\sqrt{1+q^2}} \equiv 2A(q),$$
$$\int_0^\infty xK_0\left(\frac{x}{2}\right)K_1\left(\frac{x}{2}\right)J_1(qx)dx = \frac{2\,\mathrm{arcsh}\,q}{\sqrt{1+q^2}} \equiv 2qA(q),$$
$$\int_0^\infty K_1^2\left(\frac{x}{2}\right)\left[J_1(qx)-\frac{qx}{2}\right] = 2\left[q-\sqrt{1+q^2}\,\mathrm{arcsh}\,q\right] \equiv -2q(1+q^2)A(q)+2q.$$

Let me show how to calculate these integrals one by one. Let me begin with

$$\begin{aligned}
F_{\mathbf{q}}^{(1,1)} &= \int \frac{d^2\mathbf{x}}{(4\pi)^2}e^{-i\mathbf{qx}}\left(\frac{x_2}{x}\,\mathrm{sh}\,\frac{x_1}{2}K_1\frac{x}{2}\right)^2\\
&= \sum_{\pm}\int \frac{d^2\mathbf{x}}{(8\pi)^2}e^{-i\mathbf{qx}}\left(\frac{x_2}{x}\right)^2(e^{\pm x_1}-1)K_1^2\frac{x}{2}\\
&= \mathrm{Re}\int_0^\infty \frac{xdx}{16\pi}\frac{\partial_{q_2}}{x}\left[\hat{p}_2J_1(px)-\hat{q}_2J_1(qx)\right]K_1^2\frac{x}{2}\\
&= \frac{1}{8\pi}\,\mathrm{Re}\,\partial_2\left[q_2(1+q^2)A(q)-p_2(1+p^2)A(p)\right]\\
&= \frac{1}{8\pi}\left(\left[\hat{q}_1^2(1+q^2)+q_2^2\right]A(q)+\hat{q}_2^2-\mathrm{Re}\left\{\left[\hat{p}_1^2(1+p^2)+p_2^2\right]A(p)+\hat{p}_2^2\right\}\right).
\end{aligned}$$

where I have introduced vector $\mathbf{p}=(q_1+i,q_2)$. The following two are similar.

$$\begin{aligned}
F_{\mathbf{q}}^{(0,2)} &= \int \frac{d^2\mathbf{x}}{(4\pi)^2}e^{-i\mathbf{qx}}\left(\mathrm{ch}\,\frac{x_1}{2}K_0\frac{x}{2}+\frac{x_1}{x}\,\mathrm{sh}\,\frac{x_1}{2}K_1\frac{x}{2}\right)^2\\
&= \sum_{\pm}\int \frac{d^2\mathbf{x}}{(8\pi)^2}e^{-i\mathbf{qx}}\left((e^{\pm x_1}+1)K_0^2\frac{x}{2}\pm 2\frac{x_1}{x}e^{\pm x_1}K_0\frac{x}{2}K_1\frac{x}{2}+\left(\frac{x_1}{x}\right)^2(e^{\pm x_1}-1)K_1^2\frac{x}{2}\right)\\
&= \mathrm{Re}\int_0^\infty \frac{xdx}{16\pi}\left(\left[J_0(px)+J_0(qx)\right]K_0^2\frac{x}{2}+2i\frac{\partial_{q_1}}{x}J_0(px)K_0\frac{x}{2}K_1\frac{x}{2}+\frac{\partial_{q_1}}{x}\left[\hat{p}_1J_1(px)-\hat{q}_1J_1(qx)\right]K_1^2\frac{x}{2}\right)\\
&= \frac{1}{8\pi}\,\mathrm{Re}\left\{A(p)+A(q)-2ip_1A(p)+\partial_1\left[q_1(1+q^2)A(q)-p_1(1+p^2)A(p)\right]\right\}\\
&= \frac{1}{8\pi}\left(\left[\hat{q}_2^2(1+q^2)+1+q_1^2\right]A(q)+\hat{q}_1^2-\mathrm{Re}\left\{\left[\hat{p}_2^2(1+p^2)-1+2ip_1+p_1^2\right]A(p)+\hat{p}_1^2\right\}\right).
\end{aligned}$$

The same way but with another sign is done the next one.

$$\begin{aligned}
F_{\mathbf{q}}^{(2,0)} &= \int \frac{d^2\mathbf{x}}{(4\pi)^2}e^{-i\mathbf{qx}}\left(\mathrm{ch}\,\frac{x_1}{2}K_0\frac{x}{2}-\frac{x_1}{x}\,\mathrm{sh}\,\frac{x_1}{2}K_1\frac{x}{2}\right)^2\\
&= \frac{1}{8\pi}\,\mathrm{Re}\left\{A(p)+A(q)+2ip_1A(p)+\partial_1\left[q_1(1+q^2)A(q)-p_1(1+p^2)A(p)\right]\right\}\\
&= \frac{1}{8\pi}\left(\left[\hat{q}_2^2(1+q^2)+1+q_1^2\right]A(q)+\hat{q}_1^2-\mathrm{Re}\left\{\left[\hat{p}_2^2(1+p^2)-1-2ip_1+p_1^2\right]A(p)+\hat{p}_1^2\right\}\right).
\end{aligned}$$

The next one is also very similar to $F_{\mathbf{q}}^{(0,2)}$. Functions $F_{\mathbf{q}}^{(0,2;1,1)}$ and $F_{\mathbf{q}}^{(2,0;1,1)}$ done similarly.

$$
\begin{aligned}
F_{\mathbf{q}}^{(0,2;2,0)} &= \int \frac{d^2\mathbf{x}}{(4\pi)^2} e^{-i\mathbf{qx}} \operatorname{ch}^2 \frac{x_1}{2} K_0^2 \frac{x}{2} - \left(\frac{x_1}{x}\right)^2 \operatorname{sh}^2 \frac{x_1}{2} K_1^2 \frac{x}{2} \\
&= \frac{1}{8\pi} \operatorname{Re}\left\{A(p) + A(q) - \partial_1 \left[q_1(1+q^2)A(q) - p_1(1+p^2)A(p)\right]\right\} \\
&= \frac{1}{8\pi} \left(\left[-\hat{q}_2^2(1+q^2) + 1 - q_1^2\right] A(q) - \hat{q}_1^2 - \operatorname{Re}\left\{\left[-\hat{p}_2^2(1+p^2) - 1 - p_1^2\right] A(p) - \hat{p}_1^2\right\}\right).
\end{aligned}
$$

Altogether, required functions are

$$
\begin{aligned}
8\pi F_{\mathbf{q}}^{(0,2)} &= \left[\hat{q}_2^2 + q^2 + 1\right] A(q) + \hat{q}_1^2 && - \operatorname{Re}\left\{\left[\hat{p}_2^2 + p^2 - 1 + 2ip_1\right] A(p) + \hat{p}_1^2\right\}, \\
8\pi F_{\mathbf{q}}^{(2,0)} &= \left[\hat{q}_2^2 + q^2 + 1\right] A(q) + \hat{q}_1^2 && - \operatorname{Re}\left\{\left[\hat{p}_2^2 + p^2 - 1 - 2ip_1\right] A(p) + \hat{p}_1^2\right\}, \\
8\pi F_{\mathbf{q}}^{(1,1)} &= \left[\hat{q}_1^2 + q^2\right] A(q) + \hat{q}_2^2 && - \operatorname{Re}\left\{\left[\hat{p}_1^2 + p^2\right] A(p) + \hat{p}_2^2\right\}, \\
-8\pi F_{\mathbf{q}}^{(0,2;2,0)} &= \left[\hat{q}_2^2 + q^2 - 1\right] A(q) + \hat{q}_1^2 && - \operatorname{Re}\left\{\left[\hat{p}_2^2 + p^2 + 1\right] A(p) + \hat{p}_1^2\right\}, \\
-8\pi F_{\mathbf{q}}^{(0,2;1,1)} &= \hat{q}_1\hat{q}_2(1 - A(q)) && - \operatorname{Re}\left\{\left[-\hat{p}_1\hat{p}_2 + ip_2\right] A(p) + \hat{p}_1\hat{p}_2\right\}, \\
-8\pi F_{\mathbf{q}}^{(2,0;1,1)} &= \hat{q}_1\hat{q}_2(A(q) - 1) && - \operatorname{Re}\left\{\left[\hat{p}_1\hat{p}_2 + ip_2\right] A(p) - \hat{p}_1\hat{p}_2\right\},
\end{aligned}
$$

with $A(q) = \operatorname{arcsh}(q)/q\sqrt{1+q^2}$ and $\mathbf{p} = (q_1 + i, q_2)$. That together with (A.1) sums up the answer for polarization operator.

Asymptotics of Polarization operator I rewrite an expression separating different orders in q.

$$\begin{aligned}
8\pi P(\mathbf{q}) = &(1-4\hat{q}_1^2)\hat{q}_2^2\left[(A(q)-1)-\operatorname{Re}\left\{q^2p^{-2}(A(p)-1)\right\}\right]-\\
&-4\hat{q}_1^2\hat{q}_2^2(\hat{q}_2^2-\hat{q}_1^2)\cdot\hat{q}_1^{-1}\operatorname{Re}\left\{iqp^{-2}(A(p)-1)\right\}+\\
&+\left[q^2A(q)-\operatorname{Re}p^2A(p)\right]+\\
&+\left[A(q)+\operatorname{Re}A(p)\right]+\\
&+4\hat{q}_1^2\hat{q}_2^2\left[A(q)-\operatorname{Re}A(p)\right]-\\
&-2\left(\hat{q}_1^2-\hat{q}_2^2\right)\operatorname{Re}ip_1A(p)-\\
&-4\hat{q}_1^2\hat{q}_2^2\cdot(\hat{q}_1\hat{q}_2)^{-1}\operatorname{Re}ip_2A(p).
\end{aligned}$$

with $A(q)=\operatorname{arcsh}(q)/q\sqrt{1+q^2}$ and $\mathbf{p}=(q_1+i,q_2)$.
With the help of `Mathematica` software, I find

$$P(\mathbf{q})\sim\frac{3}{16\pi q^2}+\frac{3}{64\pi q^4}-(3-2\cos 2\varphi)\frac{\ln 2q}{16\pi q^4}+\frac{\cos 4\varphi-32\cos 2\varphi}{192\pi q^4}+o\left(\frac{1}{q^4}\right),\quad q\to\infty.$$

From here it follows that

$$L_1(\mathbf{q})=\frac{1}{4}\left(2+q\frac{\partial}{\partial q}\right)P(\mathbf{q})\sim-\frac{1}{4}\frac{9}{32\pi q^4}+(3-2\cos 2\varphi)\frac{\ln 2q}{32\pi q^4}+\frac{44\cos 2\varphi-\cos 4\varphi}{4\times 96\pi q^4},\quad q\to\infty$$

$$\langle L_1(\mathbf{q})\rangle_{\hat{\mathbf{q}}}\sim\frac{-9/4+3\ln 2q}{32\pi q^4}.$$

Comment Earlier I've used the following expression for asymptotics of L_α.

$$\begin{aligned}
\langle L_1(\mathbf{q})\rangle_{\hat{\mathbf{q}}}&\sim\frac{1}{q^4}\int\frac{k_1^2(d\mathbf{k})}{(k^4+k_1^2/q^2)^2}\left\langle\frac{[\mathbf{k}\times\hat{\mathbf{q}}]^4}{(\mathbf{k}-\hat{\mathbf{q}})^4}\right\rangle_{\hat{\mathbf{q}}}=\int_0^\infty\frac{dk f(k)}{2\pi k}\left\langle\frac{\hat{k}_1^2}{(k^2+\hat{k}_1^2/q^2)^2}\right\rangle_{\hat{\mathbf{k}}}\\
&=\frac{1}{q^4}\int_0^\infty\frac{dk}{4\pi q^2}\frac{f(k)}{k^2(1/q^2+k^2)^{3/2}}\sim\frac{-9/4+3\ln 2q}{32\pi q^4},\quad q\to\infty.
\end{aligned}$$

Which somehow justifies validity of the used way to calculate asymptotics in the main text.

Small argument At $q\ll 1$ asymptotic of $P(\mathbf{q})$ is highly anisotropic.

$$P(\mathbf{q})\sim\frac{1}{8}\begin{cases}\frac{1}{\pi}, & q_2^2\gg q_1,\\ \hat{q}_1^4|q_1|^{-1/2}, & q_2^2\ll q_1,\end{cases}\qquad\sqrt{q_1^2+q_2^2}\ll 1.$$

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