PROBLEM SET #2
"Thermodynamics of Fermi gas"
due on 4/18/2021

Physics–172 / Applied Physics–272
Introduction to Solid State Physics
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Remark. For brevity, we set Boltzmann constant $k_B = 1$ to unity in that problem set.

Problem 2.1 (Entropy — **2 points**). Derive the following expression for the entropy of the free electron system.

$$S = -\int \frac{V d^3\mathbf{k}}{4\pi^3} \left[n_F \ln n_F + (1 - n_F) \ln(1 - n_F) \right],$$

where $n_F = n_F(\varepsilon_\mathbf{k} - \mu)$ is the Fermi function, and μ is the chemical potential. You may use expression for the Grand canonical ensemble free energy $\Omega = \Omega(\mu, V, T)$ as your starting point.

$$\Omega = -T \int \frac{V d^3\mathbf{k}}{4\pi^3} \ln \left[1 + \exp \frac{\mu - \varepsilon_\mathbf{k}}{T} \right], \quad \varepsilon_\mathbf{k} \equiv \frac{\hbar^2 \mathbf{k}^2}{2m}.$$

Problem 2.2 (Equation of state — **3 points**). In that problem P is the pressure defined through the Grand canonical ensemble free energy $\Omega = \Omega(\mu, V, T)$ as

$$P = -\left(\frac{\partial \Omega}{\partial V} \right)_{\mu,T}.$$

1. (**1 point**) Show that $P(\mu, T)$ is a homogeneous function of degree 5/2 in its variables μ, T. In other words, show that the following relation holds.

$$P(\lambda\mu, \lambda T) = \lambda^{5/2} P(\mu, T).$$

2. (**1 point**) Show that

$$\left(\frac{\partial P}{\partial \mu} \right)_T = n \qquad \text{and} \qquad \left(\frac{\partial P}{\partial T} \right)_\mu = s.$$

3. (**1 point**) Derive the following identity

$$E = \frac{3}{2} PV.$$

Remark. Heat capacity of real metals behaves (see Figure 1) at low temperatures as $C \sim \gamma T + \alpha T^3$, where linear–in–$T$ contribution comes from electrons and cubic term appears due to energy of lattice vibrations — energy of phonons. Naturally, phonon contributions becomes dominant in insulators and semiconductors, so it is important to include it into consideration. In the next problems we will calculate cubic contribution from electrons, and later in the class we will derive an expression for phonon heat capacity within Debye model and compare them.

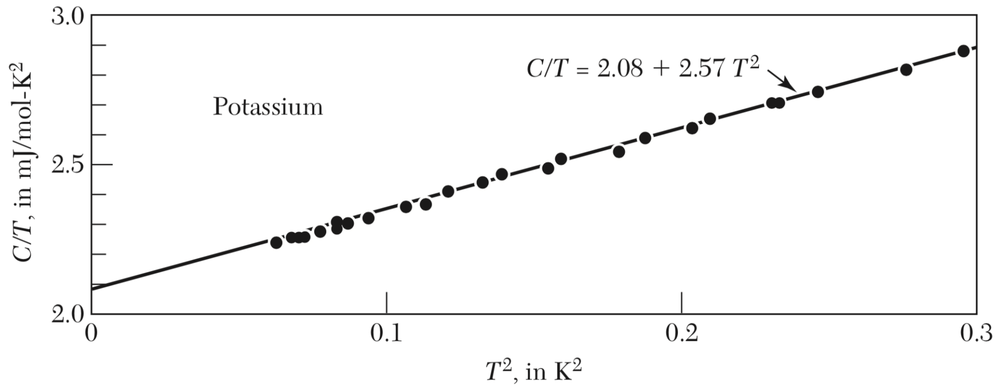

Figure 1: Experimental heat capacity values for potassium, plotted as C/T versus T^2 (borrowed from Kittel, who borrowed it from W. H. Lien and N. E. Phillips).

Problem 2.3 (Correction to the specific heat of electron gas — **4 points**)**.** Compute the T^3 contribution to the specific heat for a free electron gas. In fact, show that

$$c_V = \frac{\pi^2}{2} n \frac{T}{\epsilon_F} + A \cdot n \left(\frac{T}{\epsilon_F}\right)^3 + \ldots, \quad T \ll \epsilon_F,$$

and calculate the prefactor A.

Hint. You may find useful the Sommerfeld expansion [1, §58].

$$\int_0^\infty \frac{g(\varepsilon) d\varepsilon}{1 + \exp\frac{\varepsilon - \mu}{T}} \sim \int_0^\mu g(\varepsilon) d\varepsilon + 2 \sum_{m=0}^\infty \frac{T^{2m+2} g^{(2m+1)}(\mu)}{(2m+1)!} \int_0^\infty \frac{z^{2m+1} dz}{e^z + 1}, \quad \frac{T}{\mu} \to 0.$$

Problem 2.4 (Sommerfeld expansion in 2D — **3 points**)**.** Applying the magnetic field in the z-direction leaves the x-y plane to respond to the field as a *two-dimensional electron gas* (2DEG).

1. (**1 point**) Derive the relation between electron density n and Fermi wavevector k_F and calculate the density of states $g(\epsilon)$ of the free electron (there is no magnetic field) system in two-dimensions.

2. (**1 point**) Show that except for the $T = 0$ term, all other terms in the low temperature expansion (a.k.a. the Sommerfeld expansion) of the electron density $n(\mu, T)$ vanish.

3. (**1 point**) Show that

$$\mu + T \ln\left(1 + e^{-\mu/T}\right) = \epsilon_F$$

Explain how this formula reconciles with Sommerfeld expansion you've derived above.

References

[1] L. D. Landau and E. M. Lifshitz. *Statistical Physics*, volume 5. Elsevier, 2013.