PROBLEM SET #4

"Electrons in metals"

due on 5/2/2021

Physics–172 / Applied Physics–272

Introduction to Solid State Physics

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Problem 4.1 (Screened potential — **6 points**)**.** Find physical solutions (i.e. vanishing at infinity) of the following problems. These problems could be tackled by Fourier transform method. Alternatively, you can guess the answer and check that it satisfies the equation.

1. (**1 point**) Poisson equation in $D = 3$ dimensions

$$-\nabla^2\phi(\mathbf{r}) = 4\pi\delta(\mathbf{r}). \qquad \mathbf{r} = (x, y, z).$$

Hint. Integral form of a delta function.

$$\delta(\mathbf{r}) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{r}}.$$

2. (**1 point**) Poisson equation in $D = 2$ dimensions

$$-(\partial_x^2 + \partial_y^2)\phi(\boldsymbol{\rho}) = 4\pi\delta(\boldsymbol{\rho}), \qquad \boldsymbol{\rho} = (x, y).$$

This problem requires introduction of a regularization of some sort.

3. (**1 point**) Screened Poisson equation in $D = 3$ dimensions

$$(-\nabla^2 + \lambda^{-2})\phi(\mathbf{r}) = 4\pi\delta(\mathbf{r}).$$

4. (**1 point**) Screened Poisson equation in $D = 2$ dimensions.

$$(-\nabla^2_{x,y} + \lambda^{-2})\phi(\boldsymbol{\rho}) = 4\pi\delta(\boldsymbol{\rho}), \qquad \boldsymbol{\rho} = (x, y).$$

Exact answer could be expressed through Macdonald function of zeroth order $K_0(z)$, however, we only ask you to find asymptotic behavior of $\phi(\rho)$ at $\rho \ll \lambda$ and $\rho \gg \lambda$.

5. (**2 points**) Three–dimensional Poisson equation screened in two–dimensional plane.

$$(-\nabla^2_{x,y,z} + 2\lambda^{-1}\delta(z))\phi(\boldsymbol{\rho}, z) = 4\pi\delta(\boldsymbol{\rho})\delta(z), \qquad \boldsymbol{\rho} = (x, y). \tag{4.1.1}$$

Exact answer for $\phi(\boldsymbol{\rho}, z = 0)$ could be expressed through Neumann $N_0(z)$ and Struve functions $H_0(z)$, however, we only ask you to find asymptotic behavior of $\phi(\rho, z = 0)$ at $\rho \ll \lambda$ and $\rho \gg \lambda$. Can you predict what it is going to be like without calculation?

Remark. To understand where equation (4.1.1) comes from, you can imagine that you have a point charge, which lies within a plane of screening media, but half–space above and below it is empty. Another system where such problem appears is a description of vortex in a thin superconductor.

Problem 4.2 (Friedel oscillations — **4 points**). Calculate local density of states (LDOS)

$$\rho(x) = 2 \sum_{k<k_F} |\psi_k(x)|^2 ,$$

where $\psi_k(x)$ are one–particle wavefunctions with wavevector k and factor 2 comes from spin, for the system of

1. (**1 point**) Fermions living on a half–line $x > 0$ with impenetrable wall $\psi(+0) = 0$.
2. (**3 points**) Fermions on a line with scattering center $V(x) = \frac{\hbar^2 q}{m}\delta(x)$, assume $x^{-1} \ll q \ll k_F$.

You may want to consider fermions in box of finite size L, and then take a limit $L \to \infty$, $N \to \infty$ such that $n = N/L = \text{const}$. Express your answer in terms of Fermi wavevector k_F.

Problem 4.3 (Helicon — **3 points**). There exists in metals a unique mode of propagation of electromagnetic waves (discovered by R. Bowers *et al*) that is intimately related to the Hall effect. Suppose we place a metal in a uniform magnetic field $\mathbf{H} = H\hat{z}$ (perpendicular to the surface of the metal) and shine circularly polarized light on its surface $\mathbf{E} = \operatorname{Re} E_0(\hat{x} + i\hat{y})e^{ikz-i\omega t}$ (light propagates along the direction of magnetic field).

1. (**1 point**) Derive relation between current $\mathbf{j} = \hat{\sigma}(\omega)\mathbf{E}$ and electric field inside the metal within Drude approximation.

$$\frac{d\mathbf{p}}{dt} = -\frac{\mathbf{p}}{\tau} - |e|\mathbf{E} - \frac{|e|}{m}\left[\frac{\mathbf{p}}{c} \times \mathbf{H}\right], \qquad \mathbf{j} = -\frac{|e|n}{m}\mathbf{p}.$$

   Express your answer in terms of Drude conductivity σ_0 and cyclotron frequency ω_c.

2. (**1 point**) Using Maxwell equations find dispersion relation $k^2c^2 = \varepsilon(\omega)\omega^2$ for such waves, i.e. determine $\varepsilon(\omega)$. Express your answer in terms of plasma frequency ω_p and cyclotron frequency ω_c.

   (2a) Assume that $\omega_c\tau \gg 1$. At what frequencies light propagates inside the metal?

3. (**1 point**) Consider a very pure metal with resistivity $\rho \simeq 10^{-9}$ $\Omega\cdot$cm and typical carrier density $n \simeq 10^{23}$ cm^{-3} that we have placed in strong magnetic field of magnitude $B = 5$ T. Provide numerical estimates for typical values of cyclotron frequency ω_c, plasma frequency ω_p and mean–free time τ.

   (3a) Show that in the regime of large field $\omega_c\tau \gg 1$ and low frequency $\omega \ll \omega_c$, dispersion relation reduces to

$$\omega = \frac{\omega_c}{\omega_p^2}k^2c^2.$$

   This low frequency wave is known as *helicon*, and it is observed in many metals.