PROBLEM SET #3

"Magnetism of electron gas"

due on 5/5/2023

Physics–172 / Applied Physics–272

Introduction to Solid State Physics

Spring quarter, 2023

Instructor: *Aharon Kapitulnik* — aharonk@stanford.edu
Class Assistant: *David Saykin* — saykind@stanford.edu
Departments of Applied Physics and of Physics, Stanford University

Remark. For brevity, we set Boltzmann constant $k_B = 1$ to unity in this problem set.

Problem 3.1 (Pauli paramagnetism — **6 points**)**.** Let's consider electron gas that interacts with magnetic field via Zeeman mechanism only (orbital field could be considered separately)

$$\hat{\mathcal{H}} = \frac{\hat{p}^2}{2m} - m_z H, \qquad \hat{m}_z = -2\mu_B \hat{s}_z = -\mu_B \hat{\sigma}_z, \quad \mu_B = \frac{|e|\hbar}{2mc}.$$

Compute magnetic susceptibility in the $T \ll E_F$ limit and small enough magnetic field (what it should be compared to for $D = 2$? for $D = 3$?). For the sake of exercise, we ask you to do the calculation using canonical ensemble free energy $F = \Omega + \mu N$.

$$\mathbf{m} = -\frac{1}{V}\left(\frac{\partial F}{\partial \mathbf{H}}\right)_{N,V,T}.$$

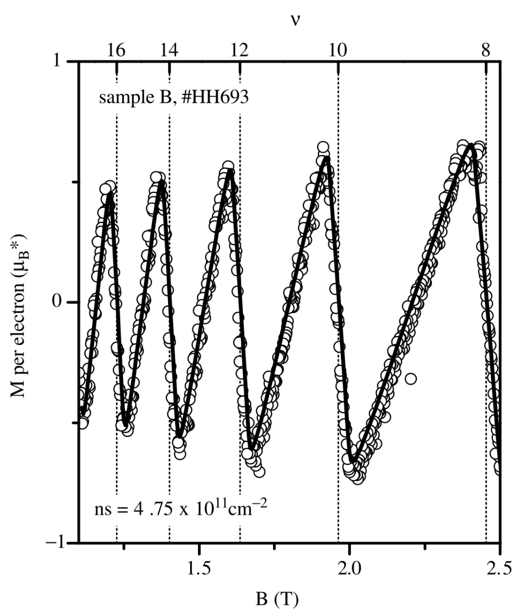

Figure 1: De Haas—van Alphen effect in GaAs [1, 2], $T = .3$ K, $n = 4.75{\times}10^{11}$ cm^{-2}, $m^* = .067 m_e$.

Problem 3.2 (Landau diamagnetism of 2DEG — **12 points**). Calculate orbital magnetic moment $\mathbf{m}$ of two–dimensional electron gas assuming the following hierarchy of scales[1]

1. (**4 points**) $\mu_B^* H \ll T \ll \epsilon_F$ (orbital diamagnetism).
2. (**6 points**) $T = 0$, $\mu_B^* H \ll \epsilon_F$ (quantum oscillations)
3. (**2 points**) Compare your answer for magnetization with measurements of dHvA in GaAs shown on Figure 1 [1, 2].

You most probably want to use Grand canonical free energy as your starting point

$$\Omega = -2T \sum_{n=0}^{\infty} g_n \ln \left[1 + \exp \frac{\mu - \varepsilon_n}{T} \right]. \qquad \mathbf{m} = -\frac{1}{S} \left(\frac{\partial \Omega}{\partial \mathbf{H}} \right)_{\mu,S,T}.$$

where g_n is a degeneracy of ε_n and factor of 2 comes from spin.

Hint. Remember when you can apply Euler–Maclaurin and Poisson summation formulas.

$$\sum_{n=0}^{\infty} F\left(n + \frac{1}{2}\right) = \int_0^{\infty} F(n) dn + \frac{F'(0)}{24} + \dots$$

$$\sum_{n=0}^{\infty} F\left(n + \frac{1}{2}\right) = \int_0^{\infty} F(n) dn + 2\,\mathrm{Re} \sum_{k=1}^{\infty} (-)^k \int_0^{\infty} e^{i2\pi k x} F(x)\, dx.$$

Remark. In **problem set #1** we have learned about quantum mechanism that holds atoms in a lattice – *exchange interaction.* It so happens that the very same mechanism is responsible for the existence of magnets. In a Hydrogen molecule there is an energy difference between state that corresponds to a total spin of two electron being $S = 0$ or $S = 1$. That allows to write down effective Hamiltonian in terms of spins of the electrons.

$$\hat{\mathcal{H}} = -\sum_{i,j} J_{i,j} \hat{\mathbf{S}}_i \cdot \hat{\mathbf{S}}_j$$

Constant J is called exchange integral, it is proportional to the difference $E_a - E_s$. In the next problem we will solve the simplest variation of such problem — Cluster Ising model — and discover the mechanism of ferromagnetism.

Problem 3.3 (Ferromagnetism — **8 points**). Consider a system of N Ising spins $s_i = \pm$ that have all–to–all ferromagnetic ($J > 0$) interaction.

$$E = -\frac{J}{2N} \sum_{i \neq j} s_i s_j - H \sum_{i=1}^{N} s_i,$$

Here J is the exchange interaction constant and H is the external magnetic field ($\mu_B = 1$). Such system exhibits ferromagnetic transition: at the temperature below Curie temperature $T < T_c$ all spins have the same value, while for $T > T_c$ there is no net magnetization. To demonstrate that, we ask you to calculate average spin value at finite temperature T in the limit $N \to \infty$.

$$m \equiv \frac{1}{N} \sum_{i=1}^{N} \langle s_i \rangle = \langle s_1 \rangle, \qquad \langle s_1 \rangle = \frac{1}{Z} \sum_{\{s_i\}} s_1 \exp\left[-\frac{E}{T} \right].$$

We ask you to do it in two different ways.

[1] Remember that Bohr magneton $\mu_B = 58\ \mu\mathrm{eV/T} = 0.67\ \mathrm{K/T}$.

1. (**2 points**) Find average spin $m = m(T, H \to +0)$ within *mean–field approximation* (you assume that all spins except one are already equal to their average value m and solve effective problem for the last spin, condition of self–consistency defines the value of m).

2. (**6 points**) Calculate free energy $F = F(T, H)$ exactly (in thermodynamic limit $N \to \infty$)

$$F = -T \lim_{N\to\infty} \ln Z, \qquad Z(T, H) = \sum_{\{s_i\}} \exp\left[-\frac{E(\{s_i\}, H)}{T}\right]$$

and determine the magnetization as function of temperature and external field.

$$m = -\frac{1}{N}\left(\frac{\partial F}{\partial H}\right)_{N,T}$$

Here are detailed steps that should help you to solve it.

(a) Rewrite the energy in terms of total spin $S = \sum_{i=1}^{n} s_i$ and decouple S^2 using Hubbard–Stratonovich transformation.

(b) Transform exponent of the sum of spins to the product of the exponents. You should end up with the following expression.

$$Z = \sqrt{\frac{NJ}{2\pi T}} \int_{-\infty}^{\infty} \exp\left[-N\frac{J\mu^2}{2T} + N \ln\left(2\cosh\frac{H + J\mu}{T}\right)\right] d\mu.$$

(c) Compute integral over effective magnetization μ (dummy variable you have introduced during Hubbard–Stratonovich) in the $N \to \infty$ limit using Laplace method (a.k.a. real version of saddle point approximation).

Find critical temperature T_c and net magnetization, make a sketch of $m = m(T, H \to +0)$. Which order transition is this?

References

[1] M. P. Schwarz, D. Grundler, H. Rolff, M. Wilde, S. Groth, Ch. Heyn, and D. Heitmann. De Haas–van Alphen effect in a two-dimensional electron system. *Physica E: Low-dimensional Systems and Nanostructures*, 12(1):140–143, 2002. Proceedings of the Fourteenth International Conference on the Ele ctronic Properties of Two-Dimensional Systems.

[2] M. P. Schwarz, M. A. Wilde, S. Groth, D. Grundler, Ch. Heyn, and D. Heitmann. Sawtooth-like de Haas–van Alphen oscillations of a two-dimensional electron system. *Phys. Rev. B*, 65:245315, Jun 2002.

[3] L. D. Landau and E. M. Lifshitz. *Statistical Physics*, volume 5. Elsevier, 2013.

[4] P. D. Grigoriev and I. D. Vagner. The de Haas-van Alphen effect in two-dimensional metals. *arXiv:cond-mat/0009409*, 2000.