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MASTER'S THESIS

Absolute Poisson ratio of two–dimensional crystalline membranes

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ABSTRACT

Fundamental theory of elasticity of two–dimensional materials is hard to develop since the interaction between phonons that is essential for the stability of a two–dimensional crystal is strongly non–linear, thus it could not ever be fully comprehended analytically, and is hard to study numerically. I develop perturbation theory in order to describe the universal anomalous Hooke's law and the auxetic Poisson ratio in two–dimensional crystalline membranes. I find that the actual critical indices may dramatically differ from the values given by the self–consistent screening approximation.

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Chapter 1

Introduction

1.1 Motivation

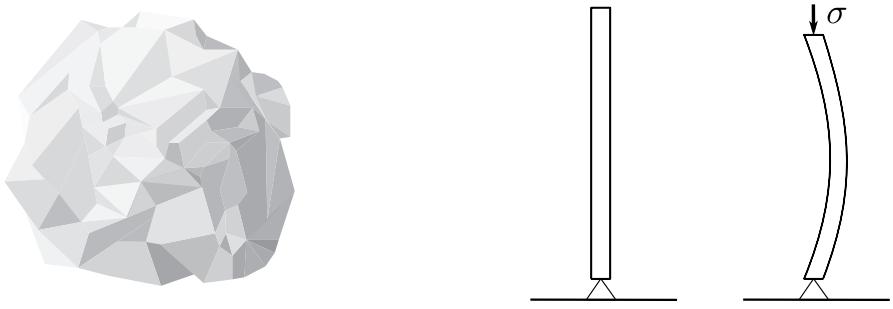
Theory of elasticity of two–dimensional crystalline membranes as we know it today has been developing since the 1980s [1] and is able to qualitatively describe some unique two–dimensional elastic phenomena such as crumpling and buckling [2], yet there are many qualitative and quantitative problems unsolved. Such theory mostly served to describe polymerized membranes in biological matter (e.g. red blood cells), however recently interest in that field of study has been revived with the experimental observation of graphene — a single atom layer of carbon. That is exactly the application I would keep in mind below.

Most prominent and curious phenomena of elastic physics of graphene are anomalous Hooke's law and negative Poisson ratio. In Figure 2a you can see the experimentally measured stress σ versus strain ξ dependence of a free–hanging graphene sheet [3]. That dependence is believed to be described by power law

$$\delta\xi = \frac{L_\sigma - L_{\sigma=0}}{L_{\sigma=0}} \propto \frac{\sigma}{Y_0}\left(\frac{\sigma}{\sigma_*}\right)^{\alpha-1}, \quad \sigma \ll \sigma_*. \tag{1.1}$$

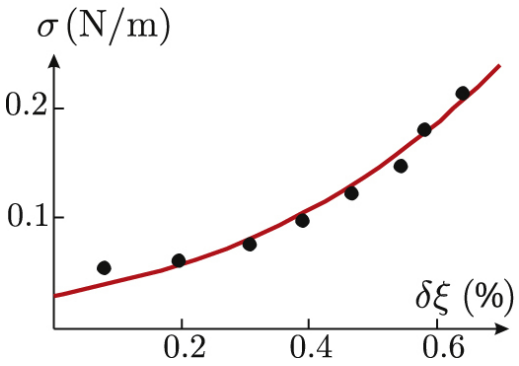
Here $Y_0 \simeq 22$ eV·Å^{-2} is the (bare) graphene Young modulus and $\sigma_* \simeq .5$ N·m^{-1} is the characteristic scale of the non–linear regime. Exponent $\alpha > 0$ is not known exactly, but several analytical approximations and numerical studies suggest that its value lies in the range $.6 \lesssim \alpha_{\text{clean}} \lesssim .7$ for the clean membranes and $\alpha_{\text{dis}} \lesssim .3$ for disordered membranes (e.g. graphene with dislocations).

Poisson ratio (PR) is defined as the strain ratio in perpendicular directions when stress is

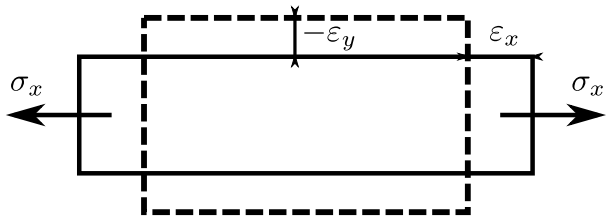


(a) Crumpled piece of paper. (b) Buckling transition.

Figure 1: Illustration of crumpling and buckling phenomena.



(a) Experimental evidence of anomalous Hooke's law $\alpha_{\text{fit}} \approx .1$ in graphene [8].



(b) Definition of Poisson ratio.

Figure 2: Unusual elastic phenomena peculiar to two–dimensional membranes.

applied only along one direction (see Figure 2b). In conventional linear regime it is given by material dependent lame coefficients and is usually positive, which means that stretching in one direction leads to shrinking in another.

$$\nu_0 = -\frac{\varepsilon_y}{\varepsilon_x} = \frac{\lambda}{2\mu + (D-1)\lambda}.$$

For graphene $\nu_0 \approx .1$. However, it so happens that in the non–linear regime (1.1) the Poisson ratio becomes a universal number for all two–dimensional crystalline surfaces, which is believed to be negative [4]. Also in such a regime, in addition to the absolute Poisson ratio, a differential one is defined.

$$\nu = -\frac{\xi_y(\sigma_x, 0) - \xi_y(0,0)}{\xi_x(\sigma_x, 0) - \xi_x(0,0)}, \qquad \nu^{\text{diff}} = -\left.\frac{\partial \xi_y(\sigma_x, \sigma)/\partial \sigma_x}{\partial \xi_x(\sigma_x, \sigma)/\partial \sigma_x}\right|_{\sigma_x = \sigma}.$$

Universal value of Poisson ratio in the regime $\sigma_L \ll \sigma \ll \sigma_*$ has been believed [5, 6] to be unprecedentedly close to $\nu_{\text{SCSA}} = -\frac{1}{3}$, however it was recently explicitly shown [7] that there is no reason to believe in the exact relation $\nu^{\text{diff}} = \nu_{\text{SCSA}}$.

Both anomalous Hooke's law and auxetic phenomena may be understood with the simple notion of ripples – small waves on the membrane surface, which are always present due to thermal fluctuations (see Figure 3). At finite temperature a free-standing surface is nearly flat but due to folds its projected area is smaller than that of the same membrane at absolute zero $T = 0$, i. e. it is shrunk in both directions. Thus, the application of external tensions will at first flatten the surface and only after that stretch the crystalline structure, which means that it is easier to spring out the membrane at small stresses. Also such initial thermal decrease of projected area is naturally isotropic, so flattening by pulling in any direction leads to expansion in both, i.e. negative Poisson ratio.

1.2 Problem statement

In my thesis I am going to study the elastic properties of a two–dimensional crystalline membrane in the non-linear (universal) stress regime $\sigma \ll \sigma_*$. In order to do that, at first I am going to derive a model of a 2D membrane under finite stress (Chapter 2) and develop a controlled perturbation theory (Chapter 3) in order to obtain analytical expressions depicting the anomalous Hooke's law. Specifically, I am going to use perturbative approach to answer the following questions.

1. How does the critical exponent α from the anomalous Hooke's law differ from the known approximate values, namely, the self–consistent screening approximation [9]? (Section 3.3)

2. Can the value of the Poisson ratio ν in the universal regime be expressed in terms of α, or is it an independent critical index? How does it differ from its value given by the self–consistent screening approximation [10]? (Section 3.4)

I am going to reinforce my conclusions based on perturbation theory with numerical Monte Carlo simulations using the effective energy functional I have derived. In Chapter 4 I explain how the simulations are made and review and discuss existing articles reporting on numerical values of critical indices.

Appendix A contains calculations that are too lengthy to be presented in the main text.

Conclusions and a results summary are presented in Chapter 5.

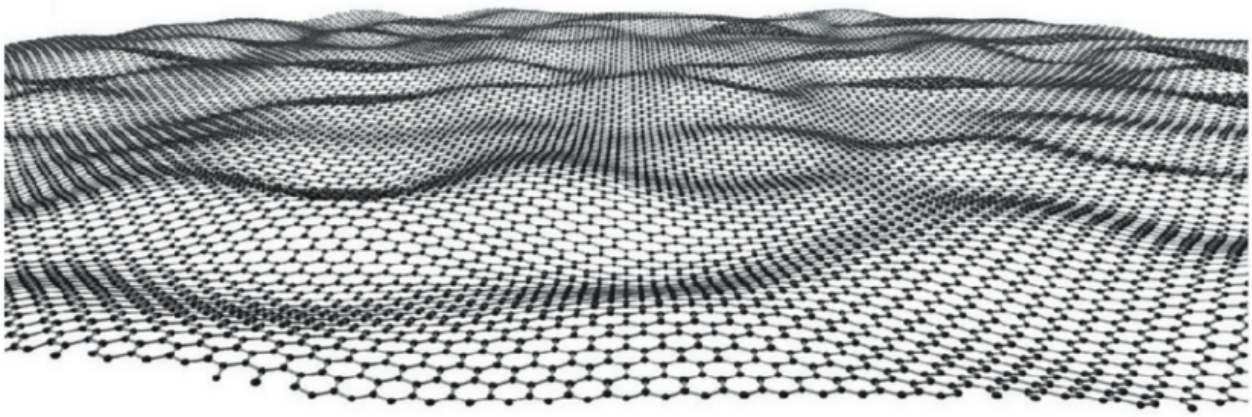


Figure 3: Graphene lattice with ripples [source].

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Chapter 2

Theoretical model

2.1 Free Energy of two–dimensional membrane

The model used to describe two–dimensional crystals (tethered surfaces) for the past several decades [11] takes into account the surface bending energy and the bonding potential [12, Ch. 6].

$$E = \bar{\varkappa} \sum_{\langle \mathbf{xy} \rangle} (1 - \mathbf{n_x} \cdot \mathbf{n_y}) + \sum_{\langle \mathbf{xy} \rangle} V\left(|\mathbf{r_x} - \mathbf{r_y}|\right).$$

Here $\mathbf{x}$ is a discrete index running through the $D = 2$–dimensional lattice nodes and the sum is taken over neighboring atoms; the normal vectors $\mathbf{n_x}$ are located at the center of an adjacent cell (see Figure 4). At zero temperature $T = 0$ the membrane is presumed to be perfectly flat, at finite temperature its profile is given by the $\mathbf{r}(\mathbf{x})$ function.

$$\mathbf{r}(\mathbf{x}) = \begin{pmatrix} x_1 + u_1(\mathbf{x}) \\ x_2 + u_2(\mathbf{x}) \\ h(\mathbf{x}) \end{pmatrix} \qquad \mathbf{x} = (x_1, x_2) \in \mathbb{R}^2.$$

In the limit of continuous media, the energy of a smooth (without self–intersections) nearly flat surface could be expanded in gradients of $\mathbf{r}(\mathbf{x})$.

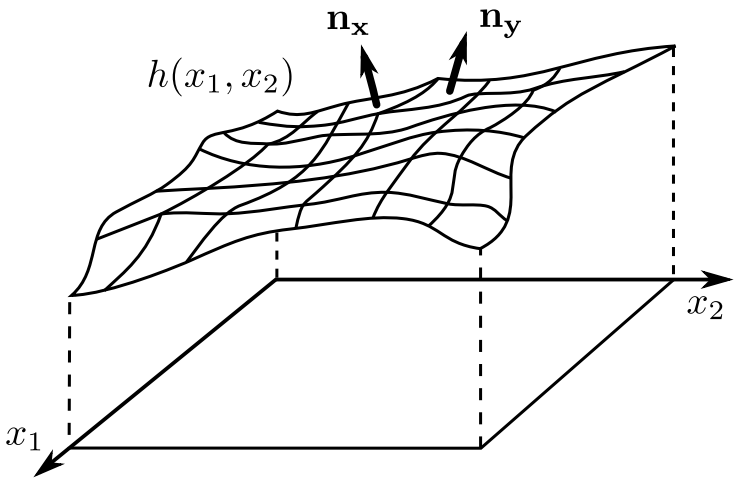


Figure 4: Parametrization of the membrane in the plane phase.

Bending energy The membrane surface is parametrized by the equation $z = h(\mathbf{x} + \mathbf{u}(\mathbf{x}))$. As will be seen below, it is enough to approximate $z \approx h(\mathbf{x})$; the normal vector is then expressed as follows.

$$\mathbf{n} = \frac{1}{\sqrt{1 + (\nabla h)^2}} \begin{pmatrix} -\partial_1 h \\ -\partial_2 h \\ 1 \end{pmatrix} \qquad h_{\mathbf{x}+\mathbf{e}_i} \approx h_{\mathbf{x}} + (\mathbf{e}_i, \nabla) h_{\mathbf{x}} + \frac{1}{2}(\mathbf{e}_i, \nabla)^2 h_{\mathbf{x}}.$$

In the equation below $\tilde{\varkappa}$ is proportional to $\bar{\varkappa}$. The proportionality coefficient depends on the lattice type and is equal to 1 for a square lattice and $1/\sqrt{3}$ for the graphene hexagonal lattice.

$$\mathbf{n}_{\mathbf{x}} \cdot \mathbf{n}_{\mathbf{x}+\mathbf{e}_i} = \sqrt{\frac{1 + (\nabla h_{\mathbf{x}})^2}{1 + (\nabla h_{\mathbf{y}})^2}} \left[1 + \frac{(\nabla h_{\mathbf{x}}, \nabla (h_{\mathbf{y}} - h_{\mathbf{x}}))}{1 + (\nabla h_{\mathbf{x}})^2} \right]_{\mathbf{y}=\mathbf{x}+\mathbf{e}_i} \approx 1 - \frac{1}{2} (\nabla_\alpha (\mathbf{e}_i \nabla) h_{\mathbf{x}})^2.$$

$$\bar{\varkappa} \sum_{\langle \mathbf{x}\mathbf{y} \rangle} (1 - \mathbf{n}_{\mathbf{x}} \cdot \mathbf{n}_{\mathbf{y}}) \approx \frac{\tilde{\varkappa}}{2} \int d^D \mathbf{x} \, (\partial_\alpha \partial_\beta h_{\mathbf{x}})^2 = \frac{\tilde{\varkappa}}{2} \int d^D \mathbf{x} \, \left[(\nabla^2 h_{\mathbf{x}})^2 - 2 \det (\partial_\alpha \partial_\beta h_{\mathbf{x}}) \right].$$

The last two terms are the mean and Gaussian curvature; the second term may be omitted since it is a perfect derivative and boundary contributions are negligible in the thermodynamic limit.

Potential energy In the same way, the potential energy can be expanded in displacement gradients.

$$V\left(|\mathbf{r}_{\mathbf{x}} - \mathbf{r}_{\mathbf{y}}|\right) = V(0) + \partial_\mu V(0) \left[(\mathbf{e}_i, \nabla) r^\mu_{\mathbf{x}} + \frac{1}{2} (\mathbf{e}_i, \nabla)^2 r^\mu_{\mathbf{x}} + \dots \right] + \dots$$

After averaging over lattice basis vectors I obtain the expression (2.1) with coefficients t, $\tilde{\lambda}$, $\tilde{\mu}$ having some tensor structure. This happens because, generally, couplings like $(\partial_\alpha r_\alpha)^2$ and $(\partial_\alpha r_\beta)(\partial_\beta r_\alpha)$ and other quartic terms are also possible; however, in the text below full rotational invariance is assumed for simplicity.

Continuous limit Altogether, energy is written as

$$\mathcal{F}[\mathbf{r}] = \frac{1}{2} \int d^D \mathbf{x} \left[\tilde{\varkappa} \left(\nabla^2 h \right)^2 + t \left(\partial_\alpha r_\beta \right)^2 + \frac{\tilde{\mu}}{2} \left(\partial_\alpha \mathbf{r} \cdot \partial_\beta \mathbf{r} \right)^2 + \frac{\tilde{\lambda}}{4} \left(\partial_\alpha r_\beta \right)^4 \right]. \tag{2.1}$$

The sign of t governs the transition between the crumpled and plane phases, as may be seen from a mean–field consideration [1, 13]. For my purposes the membrane could be considered deep in the plane phase ($t < 0$), then it is convenient to rescale $\mathbf{r} \mapsto \xi_0 \mathbf{r}$ and introduce $\varkappa = \xi_0^2 \tilde{\varkappa}$,

$\mu = \xi_0^4\tilde{\mu}$, $\lambda = \xi_0^4\tilde{\lambda}$ with $\xi_0^2 = -t/(\tilde{\mu} + D\tilde{\lambda}/2)$.

$$\mathcal{F}[\mathbf{r}] = \frac{1}{2}\int d^D\mathbf{x}\left[\varkappa(\nabla^2 h)^2 + \frac{\mu}{2}(\partial_\alpha\mathbf{r}\cdot\partial_\beta\mathbf{r} - \delta_{\alpha\beta})^2 + \frac{\lambda}{4}(\partial_\alpha\mathbf{r}\cdot\partial_\alpha\mathbf{r} - D)^2\right]. \tag{2.2}$$

2.1.1 External stress

In the presence of external stress, the free energy should obtain the $-\sigma_{\alpha\beta}\delta\partial_\alpha r_\beta$ addition [1], similarly to the term $-p\delta V$ in the thermodynamics of gases.

$$\mathcal{F}_{\text{tot}}[\mathbf{r}, \sigma_{\alpha\beta}] = \mathcal{F}[\mathbf{r}] + \mathcal{F}_{\text{ext}}[\mathbf{r}, \sigma_{\alpha\beta}], \qquad \mathcal{F}_{\text{ext}} = -\int d^D\mathbf{x}\,\sigma_{\alpha\beta}(\mathbf{x})\partial_\alpha r_\beta(\mathbf{x}).$$

Generally, one has to distinguish between coupling the stress directly to the strain tensor, as it is done in [14, §3], and using a more precise definition of work. As I will show below, the difference between these two options is negligible.

In this work I will only consider the case of uniform stress $\sigma_{\alpha\beta} = \text{const}(\mathbf{x})$. It is realized, for example, in the case of a suspended rectangular or circular membrane to which a uniform tension along the edge is applied. In each of these cases I can rewrite the additional term as a boundary integral.

$$\mathcal{F}_{\text{ext}} = -\sigma_{\alpha\beta}\int_\Omega d^D\mathbf{x}\,\partial_\alpha r_\beta(\mathbf{x}) = -\sigma_{\alpha\beta}\oint_{\mathbf{x}\in\partial\Omega} n_\alpha(\mathbf{x}) r_\beta(\mathbf{x}) dl.$$

I define the stretching coefficients as the ratio of the given length to the initial length (at $T = 0$) and shift the integration variable in $\mathcal{F}_{\text{tot}}$ in such a way $r_\mu = \xi_{\mu\alpha}x_\alpha + \delta r_\mu$ that the external term becomes $\mathbf{r}$–independent and thus may be excluded from the action.

$$\mathcal{F}_{\text{tot}}[\xi_{\mu\alpha}x_\alpha + \delta r_\mu, \sigma_{\alpha\beta}] = \mathcal{F}[\xi_{\mu\alpha}x_\alpha + \delta r_\mu] - L^D\xi_{\beta\alpha}\sigma_{\alpha\beta}$$

since $\delta r_\mu = 0$ at the boundary by construction. It is easier to see how that happens on the explicit examples below.

Rectangle The simplest possible case, which is considered in the rest of the paper, is a sample of rectangular shape.

$$\Omega_{\text{rect}} = \left[-\frac{L_x^0}{2}, \frac{L_x^0}{2}\right] \times \left[-\frac{L_y^0}{2}, \frac{L_y^0}{2}\right].$$

Strain tensor and stress term have the simplest possible form

$$\begin{aligned} &\xi_{xx} = \frac{L_x}{L_x^0}, \quad \xi_{yx} = 0, \\ &\xi_{xy} = 0 \qquad \xi_{yy} = \frac{L_y}{L_y^0}. \end{aligned} \qquad\qquad \mathcal{F}_{\text{ext}} = -\sigma_{xx} L_x L_y^0 - \sigma_{yy} L_y L_x^0.$$

Here L_x, L_y are the actual sizes of the membrane, i.e. the sizes of the projected area, whereas L_x^0, L_y^0 are the sizes of the $\mathbf{x}$–grid.

Circle Another way to have a uniform stress tensor is to impose a circular geometry.

$$\Omega_{\text{circ}} = \{\mathbf{x} : |\mathbf{x}| < R_0\}.$$

Similarly, the actual R is a function of stress.

$$\begin{aligned} &\xi_{rr} = \frac{R}{R_0}, \quad \xi_{\varphi r} = 0, \\ &\xi_{r\varphi} = 0 \qquad \xi_{\varphi\varphi} = 0. \end{aligned} \qquad\qquad \mathcal{F}_{\text{ext}} = -\sigma_{rr} \pi R R_0.$$

General approach In the general case of non–uniform stress, similar expressions may be obtained in the thermodynamic limit. Following the idea of the paper [7], I am going to work in the setting with a fixed spatial derivative $\langle \partial_\alpha r_\beta \rangle = m_{\alpha\beta}(\mathbf{x})$ rather than a given stress tensor $\sigma_{\alpha\beta}(\mathbf{x})$. I define the Legendre transform of the functional $\tilde{\mathcal{F}}_{tot}$ according to the usual rule

$$\tilde{\mathcal{F}}_{\text{tot}}[m_{\alpha\beta}] = \left[\mathcal{F}_{tot}[\sigma_{\alpha\beta}] + \int d^D\mathbf{x}\, \sigma_{\alpha\beta}(\mathbf{x}) m_{\alpha\beta}(\mathbf{x}) \right]_{\sigma_{\alpha\beta}=\sigma_{\alpha\beta}(m_{\alpha\beta})} \tag{2.3}$$

where $\sigma_{\alpha\beta}$ is expressed in terms of $m_{\alpha\beta}$ through the inverse of the relation

$$m_{\alpha\beta} = \langle \partial_\alpha r_\beta \rangle_{\mathcal{F}_{\text{tot}}} = -\frac{1}{L^D} \frac{\delta \mathcal{F}_{\text{tot}}}{\delta \sigma_{\alpha\beta}}.$$

In turn, such a functional has to satisfy

$$\sigma_{\alpha\beta} = \frac{1}{L^D} \frac{\delta \tilde{\mathcal{F}}_{\text{tot}}}{\delta m_{\alpha\beta}}.$$

In the article [7] it is stated that $\tilde{\mathcal{F}}_{\text{tot}}[m_{\alpha\beta}]$ coincides with $\mathcal{F}$ evaluated with the constraint $\langle \partial_\alpha r_\beta \rangle = m_{\alpha\beta}$, such an action can be written as

$$e^{-\mathcal{F}_1/T} = e^{-\mathcal{F}/T} \delta[\partial_\alpha r_\beta - m_{\alpha\beta}].$$

Or, equivalently, with an integral over $s_{\alpha\beta}$ that runs along the imaginary line.

$$\begin{aligned}\mathcal{F}_1[\mathbf{r}, m_{\alpha\beta}] &= -T \ln \int D[s_{\alpha\beta}] \exp\left[-\frac{\mathcal{F}}{T} + \int \frac{s_{\alpha\beta}}{T} \left(\partial_\alpha r_\beta - m_{\alpha\beta}\right) d^D\mathbf{x}\right] \\ &= -T \ln \int D[s_{\alpha\beta}] \exp\left[-\frac{\mathcal{F}_{\rm tot}[s_{\alpha\beta}]}{T} - \int \frac{s_{\alpha\beta}}{T} m_{\alpha\beta} d^D\mathbf{x}\right]\end{aligned}$$

Evaluating the integral over $s_{\alpha\beta}$ in the saddle point approximation, I come to (2.3).

2.1.2 Harmonic approximation

It was shown in the previous section that in order to describe a free–standing membrane with fixed uniform applied stress $\sigma_{\alpha\beta}$, one can consider the action

$$\mathcal{F}[\mathbf{r}] = \frac{1}{2}\int d^D\mathbf{x}\left[\varkappa(\nabla^2 h)^2 + \frac{\mu}{2}(\partial_\alpha\mathbf{r}\cdot\partial_\beta\mathbf{r} - \delta_{\alpha\beta})^2 + \frac{\lambda}{4}(\partial_\alpha\mathbf{r}\cdot\partial_\alpha\mathbf{r} - D)^2\right]. \tag{2.4}$$

with the shifted variable $\mathcal{F} = \mathcal{F}[\xi_{\mu\alpha}x_\alpha + \delta r_\mu]$ and imposed conditions

$$\delta r_{x,y}|_{\partial\Omega} = 0, \qquad \frac{\delta\mathcal{F}}{\delta\xi_{\beta\alpha}(\mathbf{x})} = \sigma_{\alpha\beta}(\mathbf{x}). \tag{2.5}$$

In order to simplify the action I make the substitution

$$\mathbf{r}(\mathbf{x}) = \begin{pmatrix}\xi_1 x_1 + u_1(\mathbf{x})\\ \xi_2 x_2 + u_2(\mathbf{x})\\ h(\mathbf{x})\end{pmatrix}$$

and expand all terms in (2.4). Let me define a strain tensor as

$$u_{\alpha\beta} = \frac{1}{2}\left(\xi_{(\alpha)}\partial_\beta u_\alpha + \xi_{(\beta)}\partial_\alpha u_\beta + \partial_\alpha\mathbf{u}\cdot\partial_\beta\mathbf{u} + \partial_\alpha h\cdot\partial_\beta h\right)$$

then I can rewrite the following terms

$$\begin{aligned}\partial_\alpha\mathbf{r}\cdot\partial_\beta\mathbf{r} &= (\xi_{(\gamma)}\delta_{\alpha\gamma} + \partial_\alpha u_\gamma + \partial_\alpha h_\gamma)(\xi_{(\gamma)}\delta_{\beta\gamma} + \partial_\beta u_\gamma + \partial_\beta h_\gamma)\\ &= (\xi_{(\alpha)}\delta_{\alpha\gamma} + \partial_\alpha u_\gamma)(\xi_{(\beta)}\delta_{\beta\gamma} + \partial_\beta u_\gamma) + \partial_\alpha h\cdot\partial_\beta h\\ &= \xi_{(\alpha)}\xi_{(\beta)}\delta_{\alpha\beta} + 2u_{\alpha\beta}.\\ \partial_\alpha\mathbf{r}\cdot\partial_\alpha\mathbf{r} &= \xi_\alpha^2 + 2u_{\alpha\alpha},\end{aligned}$$

where the Einstein summation rule is presumed as always, so that, for example, $\xi_\alpha^2 = \xi_\alpha\xi^\alpha = \xi_x^2 + \xi_y^2$.

Here I define proportional deformations ε_α in the following way supposing that $\xi_\alpha \approx 1$.

$$\varepsilon_\alpha \rightleftharpoons \frac{\xi_\alpha\xi_{(\alpha)} - 1}{2} \approx \xi_\alpha - 1 = \frac{\Delta_\alpha L}{L}, \quad \boldsymbol{\varepsilon} = \frac{1}{2}\begin{pmatrix}\xi_x^2 - 1\\ \xi_y^2 - 1\end{pmatrix}$$

Local proportional deformations for uniaxial deformations are given by the diagonal elements of the strain tensor; I denote them as $\frac{1}{2}K_\alpha$.

$$K_\alpha \rightleftharpoons 2u_{\alpha(\alpha)}, \quad \mathbf{K} = \begin{pmatrix}2u_{xx}\\ 2u_{yy}\end{pmatrix}$$

I modify expressions further.

$$\begin{aligned}
\frac{1}{4}(\partial_\alpha \mathbf{r}\cdot\partial_\beta \mathbf{r}-\delta_{\alpha\beta})^2 &= \frac{1}{4}(\xi_{(\alpha)}\xi_{(\beta)}\delta_{\alpha\beta}-\delta_{\alpha\beta}+2u_{\alpha\beta})(\xi_{(\alpha)}\xi_{(\beta)}\delta_{\alpha\beta}-\delta_{\alpha\beta}+2u_{\alpha\beta})\\
&= (\varepsilon_{(\alpha)}\delta_{\alpha\beta}+u_{\alpha\beta})(\varepsilon_{(\alpha)}\delta_{\alpha\beta}+u_{\alpha\beta})\\
&= \varepsilon_\alpha^2+2\varepsilon_{(\alpha)}u_{\alpha\alpha}+u_{\alpha\beta}^2\\
&= \varepsilon_\alpha(\varepsilon_\alpha+K_\alpha)+u_{\alpha\beta}^2.\\
\frac{1}{4}(\partial_\alpha \mathbf{r}\cdot\partial_\alpha \mathbf{r}-D)^2 &= \frac{1}{4}(\xi_\alpha^2-\delta_{\alpha\alpha}+2u_{\alpha\alpha})^2\\
&= \left(\varepsilon_\alpha+\frac{1}{2}K_\alpha\right)1_{\alpha\beta}\left(\varepsilon_\beta+\frac{1}{2}K_\beta\right)\\
&= \varepsilon_\alpha 1_{\alpha\beta}(\varepsilon_\beta+K_\beta)+(u_{\alpha\alpha})^2.
\end{aligned}$$

Here $1_{\alpha\beta}$ is a 2×2 matrix with all elements equal to unity. The sum of these terms is

$$\frac{\mu}{4}(\partial_\alpha \mathbf{r}\cdot\partial_\beta \mathbf{r}-\delta_{\alpha\beta})^2+\frac{\lambda}{8}(\partial_\alpha \mathbf{r}\cdot\partial_\alpha \mathbf{r}-D)^2=\varepsilon_\alpha\left[\mu\delta_{\alpha\beta}+\frac{\lambda}{2}1_{\alpha\beta}\right](\varepsilon_\beta+K_\beta)+\mu(u_{\alpha\alpha})^2+\frac{\lambda}{2}u_{\alpha\beta}^2.$$

So that the free energy is given by (compare with [14, (4.1)])

$$\mathcal{F}[\mathbf{u},h,\varepsilon]=\int d^D\mathbf{x}\left\{\frac{\varkappa}{2}(\nabla^2 h)^2+\mu(u_{\alpha\alpha})^2+\frac{\lambda}{2}u_{\alpha\beta}^2+\frac{1}{2}\varepsilon_\alpha M_{\alpha\beta}(\varepsilon_\beta+\overline{K_\beta})\right\},\tag{2.6}$$

where I have introduced the matrix

$$M_{\alpha\beta}\rightleftharpoons 2\mu\delta_{\alpha\beta}+\lambda 1_{\alpha\beta},\qquad M=\begin{pmatrix}2\mu+\lambda & \lambda\\ \lambda & 2\mu+\lambda\end{pmatrix}.\tag{2.7}$$

and switched K_α with their space-averaged values.

$$\overline{K}_\beta=\int\frac{d^D\mathbf{x}}{L^D}K_\beta=\int d^D\mathbf{x}\left(\partial_\beta\mathbf{u}\cdot\partial_{(\beta)}\mathbf{u}+\partial_\beta\mathbf{h}\cdot\partial_{(\beta)}\mathbf{h}\right)$$

since $\mathbf{u}(0)=\mathbf{u}(L)=0$.

Harmonic approximation In order to move on, I first consider the harmonic limit. I define the Green functions as

$$\begin{aligned}
L^{-D}\left\langle u_{\mathbf{q}}^\alpha u_{-\mathbf{q}}^\beta\right\rangle &= F_{\mathbf{q}}^{(l)}\frac{q_\alpha q_\beta}{q^2}+F_{\mathbf{q}}^{(t)}\left(\delta_{\alpha\beta}-\frac{q_\alpha q_\beta}{q^2}\right),\\
L^{-D}\left\langle h_{\mathbf{q}}h_{-\mathbf{q}}\right\rangle &= G_{\mathbf{q}}.
\end{aligned}$$

For simplicity I set $\xi_x = \xi_y = \xi$, then in the harmonic approximation [15, D2].

$$F_{\mathbf{q}}^{(l)} = \frac{1}{(2\mu+\lambda)\xi^2 + (\mu+\lambda)(\xi^2-1)} \frac{1}{q^2}$$
$$F_{\mathbf{q}}^{(t)} = \frac{1}{\mu\xi^2 + (\mu+\lambda)(\xi^2-1)} \frac{1}{q^2}$$
$$G_{\mathbf{q}} = \frac{T}{\varkappa q^4 + (\mu+\lambda)(\xi^2-1)q^2}$$

From here I see that for $q < q_\varkappa = \sqrt{\mu/\varkappa} \simeq 3$ Å^{-1} I can neglect[1] $uu \ll hh$ in the expressions for the strain tensor $u_{\alpha\beta}$ and K_α. Since $q_\varkappa$ is numerically of order the ultraviolet cutoff $\frac{\pi}{a} \simeq 3$ Å^{-1}, that is what is usually done [15].

Now it is suitable to rescale $\xi_{(\alpha)} u_\alpha \mapsto u_\alpha$, so that

$$u_{\alpha\beta} = \frac{1}{2}\left(\partial_\beta u_\alpha + \partial_\alpha u_\beta + \partial_\alpha h \cdot \partial_\beta h\right) \qquad \overline{K}_\beta = \int \frac{d^D\mathbf{x}}{L^D}\left(\partial_{(\beta)} h \cdot \partial_\beta h\right).$$

High d_c expansion Here I note that the anharmonic terms contain no smallness, which makes the problem of calculating the functional integral intractable analytically. I introduce a fictitious large parameter $d_c \gg 1$ — the size of the $\mathbf{h}$ vector. The physical case corresponds to $d_c = 1$; however, keeping arbitrary d_c helps to develop a controllable perturbation theory, which will be done in Chapter 3.

$$\mathcal{F}[\mathbf{u},\mathbf{h}] = \frac{1}{2}\int d^D\mathbf{x}\left\{\varkappa(\nabla^2\mathbf{h})^2 + 2\mu(u_{\alpha\alpha})^2 + \lambda u_{\alpha\beta}^2 + M_{\alpha\beta}\left[\left(\varepsilon_\alpha + \frac{\overline{K}_\alpha}{2}\right)\left(\varepsilon_\beta + \frac{\overline{K}_\beta}{2}\right) - \frac{\overline{K}_\alpha\overline{K}_\beta}{4}\right]\right\}.$$

The strain tensor is now written as

$$u_{\alpha\beta} = \frac{1}{2}\left(\partial_\beta u_\alpha + \partial_\alpha u_\beta + \partial_\alpha \mathbf{h} \cdot \partial_\beta \mathbf{h}\right) \qquad \overline{K}_\beta = \int \frac{d^D\mathbf{x}}{L^D}\left(\partial_{(\beta)} \mathbf{h} \cdot \partial_\beta \mathbf{h}\right).$$

[1] The conventional theory of elasticity neglects both u^2 and h^2, thus making the action Gaussian.

2.1.3 Effective action for flexural phonons

In order to proceed I want to integrate out $\mathbf{u}$—modes.

$$e^{-\mathcal{F}_{\text{eff}}[\mathbf{h}]/T} = \int D[\mathbf{u}] e^{-\mathcal{F}[\mathbf{u},\mathbf{h}]/T}.$$

I expand $u_{\alpha\beta}$ terms

$$\begin{aligned}
(u_{\alpha\beta})^2 &= \frac{1}{4}\left(\partial_\beta u_\alpha + \partial_\alpha u_\beta + \partial_\alpha \mathbf{h}\cdot\partial_\beta \mathbf{h}\right)^2 \\
&= \frac{1}{2}(\partial_\alpha u_\beta)^2 + \frac{1}{2}\partial_\alpha u_\beta \cdot \partial_\beta u_\alpha + \partial_\alpha u_\beta \cdot (\partial_\alpha \mathbf{h}\cdot\partial_\beta \mathbf{h}) + \frac{1}{4}(\partial_\alpha \mathbf{h}\cdot\partial_\beta \mathbf{h})^2 \\
(u_{\alpha\alpha})^2 &= \frac{1}{4}\left(2\partial_\alpha u_\alpha + \partial_\alpha \mathbf{h}\cdot\partial_\alpha \mathbf{h}\right)^2 \\
&= (\partial_\alpha u_\alpha)^2 + \partial_\beta u_\beta \cdot (\partial_\alpha \mathbf{h}\cdot\partial_\alpha \mathbf{h}) + \frac{1}{4}(\partial_\alpha \mathbf{h}\cdot\partial_\alpha \mathbf{h})^2
\end{aligned}$$

and divide energy into three parts

$$\mathcal{F} = \mathcal{F}_u + \mathcal{F}_{uh} + \mathcal{F}_h,$$

where $\mathcal{F}_u$ is quadratic in $\mathbf{u}$, $\mathcal{F}_{uh}$ is linear in $\mathbf{u}$ and $\mathcal{F}_h$ contains everything else.

$$\begin{aligned}
\mathcal{F}_u &= \frac{1}{2}\int d^D\mathbf{x}\left\{\mu\left[(\partial_\alpha u_\beta)^2 + \partial_\alpha u_\beta \cdot \partial_\beta u_\alpha\right] + \lambda(\partial_\alpha u_\alpha)^2\right\}. \\
\mathcal{F}_{uh} &= \int d^D\mathbf{x}\left\{\mu\partial_\alpha u_\beta \cdot (\partial_\alpha \mathbf{h}\cdot\partial_\beta \mathbf{h}) + \frac{\lambda}{2}\partial_\beta u_\beta \cdot (\partial_\alpha \mathbf{h}\cdot\partial_\alpha \mathbf{h})\right\}. \\
\mathcal{F}_h &= \frac{1}{2}\int d^D\mathbf{x}\left\{\varkappa(\nabla^2\mathbf{h})^2 + \frac{\mu}{2}(\partial_\alpha \mathbf{h}\cdot\partial_\beta \mathbf{h})^2 + \frac{\lambda}{4}(\partial_\alpha \mathbf{h}\cdot\partial_\alpha \mathbf{h})^2 + M_{\alpha\beta}\left[\dots\right]\right\}.
\end{aligned}$$

I impose periodic boundary conditions since it is much easier to work with them than with zero boundary conditions. Performing a Fourier transformation

$$\mathbf{u}(\mathbf{x}) = \int (d\mathbf{q}) e^{i\mathbf{q}\mathbf{x}}\mathbf{u}_\mathbf{q} \equiv \int \frac{d^D\mathbf{q}}{(2\pi)^D} e^{i\mathbf{q}\mathbf{x}}\mathbf{u}_\mathbf{q} = \frac{1}{L^D}\sum_\mathbf{q} e^{i\mathbf{q}\mathbf{x}}\mathbf{u}_\mathbf{q} \tag{2.8}$$

and exploiting that the $\mathbf{u}$ fields are real, $\mathbf{u}_{-\mathbf{q}} = \mathbf{u}^*_\mathbf{q}$, I come to the expressions

$$\begin{aligned}
\mathcal{F}_u &= \frac{1}{2}\int (d\mathbf{q})(u^\alpha_\mathbf{q})^*\left\{\mu q^2\delta_{\alpha\beta} + (\mu+\lambda)q_\alpha q_\beta\right\}u^\beta_\mathbf{q}. \\
\mathcal{F}_{uh} &= \int (d\mathbf{q})\int d^D\mathbf{x}\, e^{i\mathbf{q}\mathbf{x}}(\partial_\alpha \mathbf{h}\cdot\partial_\beta \mathbf{h})\left\{\mu i q_\alpha\delta_{\beta\gamma} + \frac{\lambda}{2}i q_\gamma\delta_{\alpha\beta}\right\}u^\gamma_\mathbf{q}.
\end{aligned}$$

Gaussian integration For each momentum $\mathbf{q} \neq 0$ integration over $u^\alpha_{\mathbf{q}}$ has the form [2]

$$\int \prod_{\mu,\nu} \frac{(du^\mu_{\mathbf{q}})^* \wedge du^\nu_{\mathbf{q}}}{2\pi i} \exp\left\{-\frac{1}{2T}(u^\alpha_{\mathbf{q}})^* A^{\mathbf{q}}_{\alpha\beta} u^\beta_{\mathbf{q}} + \frac{1}{2T}(B^{\mathbf{q}}_\gamma)^* u^\gamma_{\mathbf{q}} + \frac{1}{2T} B^{\mathbf{q}}_\gamma (u^\gamma_{\mathbf{q}})^*\right\} = \\ = \exp\left\{\frac{1}{2T}(B^{\mathbf{q}}_\gamma)^* (A^{\mathbf{q}})^{-1}_{\gamma\gamma'} B^{\mathbf{q}}_{\gamma'}\right\} \frac{T^D}{\det A^{\mathbf{q}}},$$

with the following matrix and vector

$$A^{\mathbf{q}}_{\alpha\beta} = \mu q^2 \delta_{\alpha\beta} + (\mu+\lambda) q_\alpha q_\beta,$$
$$B^{\mathbf{q}}_\gamma = \frac{i}{2}\int d^D\mathbf{x}\, e^{-i\mathbf{q}\mathbf{x}} (\partial_\alpha \mathbf{h}\cdot\partial_\beta\mathbf{h}) \left\{\mu q_\alpha \delta_{\beta\gamma} + \mu q_\beta \delta_{\alpha\gamma} + \lambda q_\gamma \delta_{\alpha\beta}\right\},$$

Since det is simply a number (not a function of h) it may be omitted in $\mathcal{F}_{\text{eff}}$, thus $B^+A^{-1}B$ is of interest. The inversion of the 2×2 matrix $A^{\mathbf{q}}$ is easily done in the basis of $\hat{\mathbf{q}} = \mathbf{q}/q$ and its orthogonal partner $\hat{\mathbf{q}}_\perp$ since the matrix is diagonal in it [3].

$$A^{\mathbf{q}}_{\alpha\beta} = \mu q^2 \delta_{\alpha\beta} + (\mu+\lambda) q_\alpha q_\beta = q^2\left(\mu P^\perp_{\alpha\beta} + (2\mu+\lambda) P^\parallel_{\alpha\beta}\right)$$
$$(A^{\mathbf{q}})^{-1}_{\alpha\beta} = \frac{1}{q^2}\left(\frac{1}{\mu} P^\perp_{\alpha\beta} + \frac{1}{2\mu+\lambda} P^\parallel_{\alpha\beta}\right).$$

Here $P^\parallel_{\alpha\beta} = q_\alpha q_\beta/q^2$ is the projector onto the $\mathbf{q}$ line and $P^\perp_{\alpha\beta} = \delta_{\alpha\beta} - q_\alpha q_\beta/q^2$ is the projector onto its orthogonal complement.

Altogether, the effective free energy consists of the following contributions

$$\mathcal{F}_{\text{eff}} = \mathcal{F}_h - \frac{1}{2}(B^{\mathbf{q}}_\gamma)^*(A^{\mathbf{q}})^{-1}_{\gamma\gamma'} B^{\mathbf{q}}_{\gamma'} \\ = \frac{1}{2}\int d^D\mathbf{x}\left\{\varkappa(\nabla^2\mathbf{h})^2 + M_{\alpha\beta}\left(\varepsilon_\alpha + \frac{\overline{K}_\alpha}{2}\right)\left(\varepsilon_\beta + \frac{\overline{K}_\beta}{2}\right)\right\} + \delta\mathcal{F}^{\mathbf{q}=0}_{\text{eff}} + \delta\mathcal{F}^{\mathbf{q}\neq 0}_{\text{eff}}.$$

where I singled out the interaction with $\mathbf{q}\neq 0$ coming from integration over $\mathbf{u}$ and the $\frac{\mu}{2}hh$, $\frac{\lambda}{4}hh$ terms, and the interaction at $\mathbf{q}=0$ produced by $\overline{KK}$ and the $\frac{\mu}{2}hh$, $\frac{\lambda}{4}hh$ terms in $\mathcal{F}_h$.

Momentum dependent interaction

$$\delta\mathcal{F}^{\mathbf{q}\neq 0}_{\text{eff}} = \frac{1}{8}\int'(d\mathbf{q})\int (d\mathbf{k}d\mathbf{k}') k_\alpha(-k-q)_\beta (\mathbf{h}_{\mathbf{k}}\cdot\mathbf{h}_{-\mathbf{k}-\mathbf{q}}) k'_{\alpha'}(-k'+q)_{\beta'}(\mathbf{h}_{\mathbf{k}'}\cdot\mathbf{h}_{-\mathbf{k}'+\mathbf{q}})\cdot C^{\mathbf{q}}_{\alpha\beta\alpha'\beta'}, \\ = \frac{1}{8}\int'(d\mathbf{q})\int d^D\mathbf{x}\, d^D\mathbf{x}'\, (\partial_\alpha\mathbf{h}\cdot\partial_\beta\mathbf{h})|_{\mathbf{x}}\cdot(\partial_{\alpha'}\mathbf{h}\cdot\partial_{\beta'}\mathbf{h})|_{\mathbf{x}'}\cdot e^{-i\mathbf{q}\mathbf{x}} C_{\alpha\beta\alpha'\beta'}(-i\partial_{\mathbf{x}'})\, e^{i\mathbf{q}\mathbf{x}'}. \quad (2.9)$$

From the line (2.9) it is clear that $C_{\alpha\beta\alpha'\beta'}$ is contracted with a tensor symmetric in the α, β and

[2] For $\mathbf{q}=0$ the coupling is absent and therefore the $B^+A^{-1}B$ term only contributes to the $\mathcal{F}_{\text{eff}}$ for $\mathbf{q}\neq 0$.
[3] Here I explicitly used the two-dimensionality of the setup, so that further on $D=2$.

α', β' indices; thus, the $\alpha \leftrightarrow \beta$ and $\alpha' \leftrightarrow \beta'$ permutations are allowed in the expressions below.

$$C^{\mathbf{q}}_{\alpha\beta\alpha'\beta'} = 2\mu\delta_{\alpha\alpha'}\delta_{\beta\beta'} + \lambda\delta_{\alpha\beta}\delta_{\alpha'\beta'} - \frac{2\mu q_\alpha \delta_{\beta\gamma} + \lambda q_\gamma \delta_{\alpha\beta}}{q}\left(\frac{1}{\mu}P^{\perp}_{\gamma\gamma'} + \frac{1}{2\mu+\lambda}P^{\parallel}_{\gamma\gamma'}\right)\frac{2\mu q_{\alpha'}\delta_{\beta'\gamma'} + \lambda q_{\gamma'}\delta_{\alpha'\beta'}}{q}$$

$$= \underbrace{2\mu\delta_{\alpha\alpha'}\delta_{\beta\beta'} - 4\mu P^{\parallel}_{\alpha\alpha'}P^{\perp}_{\beta\beta'} - \frac{4\mu^2}{2\mu+\lambda}P^{\parallel}_{\alpha\alpha'}P^{\parallel}_{\beta\beta'}}_{\text{tensor structure: } \#_{\alpha\alpha'}\#_{\beta\beta'}} + \underbrace{\frac{2\mu\lambda}{2\mu+\lambda}\left[\delta_{\alpha\beta}\delta_{\alpha'\beta'} - P^{\parallel}_{\alpha\beta}\delta_{\alpha'\beta'} - \delta_{\alpha\beta}P^{\parallel}_{\alpha'\beta'}\right]}_{\text{tensor structure: } \#_{\alpha\beta}\#_{\alpha'\beta'}}$$

Here I substitute $\delta_{\alpha\beta} = P^{\parallel}_{\alpha\beta} + P^{\perp}_{\alpha\beta}$ everywhere and use that $P^{\parallel}_{\alpha\alpha'}P^{\parallel}_{\beta\beta'} = q^{-4}q_\alpha q_\beta q_{\alpha'}q_{\beta'} = P^{\parallel}_{\alpha\beta}P^{\parallel}_{\alpha'\beta'}$.

$$= \underbrace{2\mu P^{\perp}_{\alpha\alpha'}P^{\perp}_{\beta\beta'} + \cancel{\frac{2\mu\lambda}{2\mu+\lambda}P^{\parallel}_{\alpha\alpha'}P^{\parallel}_{\beta\beta'}}}_{\text{tensor structure: } \#_{\alpha\alpha'}\#_{\beta\beta'}} + \underbrace{\frac{2\mu\lambda}{2\mu+\lambda}\left[P^{\perp}_{\alpha\beta}P^{\perp}_{\alpha'\beta'} - \cancel{P^{\parallel}_{\alpha\beta}P^{\parallel}_{\alpha'\beta'}}\right]}_{\text{tensor structure: } \#_{\alpha\beta}\#_{\alpha'\beta'}}$$

$$= 2\mu P^{\perp}_{\alpha\alpha'}P^{\perp}_{\beta\beta'} + \frac{2\mu\lambda}{2\mu+\lambda}P^{\perp}_{\alpha\beta}P^{\perp}_{\alpha'\beta'}. \tag{2.10}$$

For $D = 2$ this expression may be simplified further owing to the existence of the fully anti–symmetric tensor $\epsilon_{\alpha\beta}$ that allows one to present $P^{\perp}_{\alpha\beta} = p^{-2}p_\alpha p_\beta$ where $p_\alpha = \epsilon_{\alpha\beta}q_\beta$ is a vector perpendicular to $\mathbf{q}$ ($\mathbf{q} \perp \mathbf{p}$).

$$\frac{p_\alpha p_\beta}{p^2} = \frac{\epsilon_{\alpha\alpha'}\epsilon_{\beta\beta'}q_{\alpha'}q_{\beta'}}{p_\gamma p_\gamma} = \frac{(\delta_{\alpha\beta}\delta_{\alpha'\beta'} - \delta_{\alpha'\beta}\delta_{\alpha\beta'})q_{\alpha'}q_{\beta'}}{\epsilon_{\gamma\mu}\epsilon_{\gamma\nu}q_\mu q_\nu} = \frac{\delta_{\alpha\beta}q^2 - q_\alpha q_\beta}{q^2} = P^{\perp}_{\alpha\beta}$$

Therefore, $P^{\perp}_{\alpha\alpha'}P^{\perp}_{\beta\beta'} = p^{-4}p_\alpha p_\beta p_{\alpha'}p_{\beta'} = P^{\perp}_{\alpha\beta}P^{\perp}_{\alpha'\beta'}$. In other words, only for $D = 2$ can the operator $P^{\perp}$ be presented as a tensor product of two vectors.

$$P^{\perp}_{D=2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad P^{\perp}_{D=3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \neq \mathbf{v} \otimes \mathbf{w}$$

All in all, for $D = 2$ dimensional case expression (2.10) becomes

$$C^{\mathbf{q}}_{\alpha\beta\alpha'\beta'} = \frac{4\mu(\mu+\lambda)}{2\mu+\lambda}P^{\perp}_{\alpha\beta}P^{\perp}_{\alpha'\beta'} = Y_0 P^{\perp}_{\alpha\beta}P^{\perp}_{\alpha'\beta'}, \qquad Y_0 = \frac{2\mu(2\mu+D\lambda)}{2\mu+\lambda(D-1)},$$

where Y_0 is the Young modulus of a two–dimensional membrane. And since in 2D $\left|\hat{P}^{\perp}_{\mathbf{q}}\mathbf{k}\right| =$

$|[\mathbf{k}\times\hat{\mathbf{q}}]|$, with the unit vector $\hat{\mathbf{q}}=\mathbf{q}/q$, the contribution $\delta\mathcal{F}_{\rm eff}^{\mathbf{q}\neq 0}$ can be rewritten as

$$\delta\mathcal{F}_{\rm eff}^{\mathbf{q}\neq 0}=\frac{Y_0}{8}\int'(d\mathbf{q})\int(d\mathbf{k}d\mathbf{k}')[\mathbf{k}'\times\hat{\mathbf{q}}]^2[\mathbf{k}\times\hat{\mathbf{q}}]^2(\mathbf{h}_{\mathbf{k}'}\cdot\mathbf{h}_{-\mathbf{k}'+\mathbf{q}})(\mathbf{h}_{\mathbf{k}}\cdot\mathbf{h}_{-\mathbf{k}-\mathbf{q}}).$$

Momentum independent interaction Now let me get back to the $\mathbf{q}=0$ contribution. Recalling how to pass from integration to summation over discrete momenta (2.8), I modify the following expressions

$$\frac{1}{2}\int d^D\mathbf{x}\left\{\frac{\mu}{2}(\partial_\alpha\mathbf{h}\cdot\partial_\beta\mathbf{h})^2+\frac{\lambda}{4}(\partial_\alpha\mathbf{h}\cdot\partial_\alpha\mathbf{h})^2\right\}=$$
$$=\frac{1}{2}\frac{1}{4L^D}\int(d\mathbf{k}d\mathbf{k}')\left[2\mu(\mathbf{k}\cdot\mathbf{k}')^2+\lambda(kk')^2\right](\mathbf{h}_{\mathbf{k}}\cdot\mathbf{h}_{-\mathbf{k}})(\mathbf{h}_{\mathbf{k}'}\cdot\mathbf{h}_{-\mathbf{k}'})+\int'(d\mathbf{q})\ldots$$

The space-averaged anomalous deformations transform as

$$\overline{K}_\alpha=\int d^D\mathbf{x}\,(\partial_\alpha\mathbf{h}\cdot\partial_{(\alpha)}\mathbf{h})=\frac{1}{L^D}\int(d\mathbf{k})k_\alpha k_{(\alpha)}(\mathbf{h}_{\mathbf{k}}\cdot\mathbf{h}_{-\mathbf{k}}),$$

$$\frac{1}{2}\int d^D\mathbf{x}\,M_{\alpha\beta}\overline{K}_\alpha\overline{K}_\beta=\frac{1}{2}\frac{1}{L^D}\int(d\mathbf{k}d\mathbf{k}')\left[2\mu(k_x^2(k_x')^2+k_y^2(k_y')^2)+\lambda(kk')^2\right](\mathbf{h}_{\mathbf{k}}\cdot\mathbf{h}_{-\mathbf{k}})(\mathbf{h}_{\mathbf{k}'}\cdot\mathbf{h}_{-\mathbf{k}'})$$

The sum of the two contributions gives

$$\delta\mathcal{F}_{\rm eff}^{\mathbf{q}=0}=\frac{\mu}{2L^D}\int(d\mathbf{k}d\mathbf{k}')k_xk_yk_x'k_y'(\mathbf{h}_{\mathbf{k}}\cdot\mathbf{h}_{-\mathbf{k}})(\mathbf{h}_{\mathbf{k}'}\cdot\mathbf{h}_{-\mathbf{k}'})=\frac{\mu}{2L^D}\left[\int d^D\mathbf{x}\,(\partial_x\mathbf{h}\cdot\partial_y\mathbf{h})\right]^2.$$

Effective action Altogether, the effective free energy depending only on out–of–plane fluctuations reads (compare with [7, (19)])

$$\mathcal{F}_{\rm eff}[\mathbf{h}]=\frac{1}{2}\int d^D\mathbf{x}\left\{\varkappa(\nabla^2\mathbf{h})^2+M_{\alpha\beta}\left(\varepsilon_\alpha+\frac{\overline{K}_\alpha}{2}\right)\left(\varepsilon_\beta+\frac{\overline{K}_\beta}{2}\right)\right\}+\frac{\mu}{2L^D}\left[\int d^D\mathbf{x}\,(\partial_x\mathbf{h}\cdot\partial_y\mathbf{h})\right]^2+$$
$$+\frac{Y_0}{8}\int(d\mathbf{q}d\mathbf{k}d\mathbf{k}')\frac{[\mathbf{k}\times\mathbf{q}]^2[\mathbf{k}'\times\mathbf{q}]^2}{q^4}(\mathbf{h}_{\mathbf{k}}\cdot\mathbf{h}_{-\mathbf{k}-\mathbf{q}})(\mathbf{h}_{\mathbf{k}'}\cdot\mathbf{h}_{-\mathbf{k}'+\mathbf{q}}).$$

2.1.4 Balance equation

The constraint (2.5) has the physical meaning of a balance equation, and the stretching factors ε are thermodynamic variables that play the role of the volume.

$$\sigma_\alpha = \frac{\partial f}{\partial \varepsilon_\alpha}, \quad f = \frac{F}{L^2} = -\frac{T}{L^2} \ln \int D[\mathbf{h}] e^{-\mathcal{F}_{\text{eff}}[\mathbf{h},\varepsilon]/T}.$$

If the strain tensor was linearized $u_{\alpha\beta} = \frac{1}{2}(\partial_\alpha u_\beta + \partial_\beta u_\alpha)$ then, as is seen from (2.6), the equation of balance would simply read

$$\sigma^0_\alpha = M_{\alpha\beta}\varepsilon_\beta$$

with the matrix $M_{\alpha\beta}$ given by (2.7). The presence of the h^2 term in the strain tensor leads to a non–linear equation of balance [8].

$$\sigma_\alpha = M_{\alpha\beta}\left(\varepsilon_\beta + \left\langle \overline{K}_\beta \right\rangle\right), \qquad \left\langle \overline{K}_\beta \right\rangle(\varepsilon) = \frac{\int D[\mathbf{h}] \overline{K}_\beta e^{-\mathcal{F}_{\text{eff}}[\mathbf{h},\varepsilon]/T}}{\int D[\mathbf{h}] e^{-\mathcal{F}_{\text{eff}}[\mathbf{h},\varepsilon]/T}}, \tag{2.11}$$

This equation may be inverted and the $\varepsilon = \varepsilon(\sigma)$ dependence obtained.

$$\begin{pmatrix} \varepsilon_x \\ \varepsilon_y \end{pmatrix} = \frac{1}{Y_0} \begin{pmatrix} 1 & -\nu_0 \\ -\nu_0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \end{pmatrix} - \frac{1}{2} \begin{pmatrix} \left\langle \overline{K}_x \right\rangle(\sigma) \\ \left\langle \overline{K}_y \right\rangle(\sigma) \end{pmatrix} \tag{2.12}$$

where Y_0 and ν_0 are the classical expressions for Young's modulus and Poisson ratio, respectively. Because of the non–linearity, equation (2.11) may have a non–zero deformation ε even in the absence of stress $\sigma = 0$; then the true deformations would be given by $\varepsilon(\sigma) - \varepsilon(0)$.

Absolute Poisson ratio Let me now consider a membrane subjected to a uniaxial stress in the x direction $\sigma_x = \sigma$, $\sigma_y = 0$. The balance equations (2.12) become simpler and I extract an expression for the PR.

$$\nu = -\frac{\varepsilon_y}{\varepsilon_x} = \frac{\nu_0 + Y_0 \left\langle \overline{K}_y \right\rangle(\sigma, 0)/2\sigma}{1 - Y_0 \left\langle \overline{K}_x \right\rangle(\sigma, 0)/2\sigma} \approx -\frac{\left\langle \overline{K}_y \right\rangle(\sigma, 0)}{\left\langle \overline{K}_x \right\rangle(\sigma, 0)}.$$

The last approximation is done in the limit of large anomalous deformations, $\sigma \ll Y_0 \left\langle \bar{\zeta}_\alpha \right\rangle \sim Y_0 T/\varkappa$.

Differential Poisson ratio For the finite stress $\sigma_x = \sigma + \delta\sigma$, $\sigma_y = \sigma$ linear responses $\delta\varepsilon_y = -\nu^{\text{diff}}\delta\varepsilon_x$ are connected via, so called, differential Poisson ratio. From (2.12) follows

$$\nu^{\text{diff}} = -\frac{\delta\varepsilon_y}{\delta\varepsilon_x} = \left.\frac{\nu_0 + \frac{Y_0}{2}\left(\frac{\partial\langle\overline{K}_y\rangle}{\partial\sigma_x}\right)_{\sigma_y}}{1 - \frac{Y_0}{2}\left(\frac{\partial\langle\overline{K}_x\rangle}{\partial\sigma_x}\right)_{\sigma_y}}\right|_{\sigma_x=\sigma_y=\sigma} \approx -\frac{\left(\partial\left\langle\overline{K}_y\right\rangle/\partial\sigma_x\right)_{\sigma_y}}{\left(\partial\left\langle\overline{K}_x\right\rangle/\partial\sigma_x\right)_{\sigma_y}}.$$

Chapter 3

Perturbation theory

The effective action for flexural phonons derived in the previous chapter can be split into three parts.

$$\mathcal{F}^{(2)}_{\text{eff}}[\mathbf{h}] = \frac{1}{2}\int (d\mathbf{p})\left[\varkappa p^4 + \varepsilon_\alpha M_{\alpha\beta} p^2_{(\beta)}\right](\mathbf{h}_{\mathbf{p}}\cdot\mathbf{h}_{-\mathbf{p}}),$$
$$\mathcal{F}^{\mathbf{q}=0}_{\text{eff}}[\mathbf{h}] = \frac{1}{4}\int (d\mathbf{k}d\mathbf{k}')\frac{1}{L^D}\left[k^2_{(\alpha)}\frac{M_{\alpha\beta}}{2}k'^2_{(\beta)} + 2\mu k_1 k_2 k'_1 k'_2\right](\mathbf{h}_{\mathbf{k}}\cdot\mathbf{h}_{-\mathbf{k}})(\mathbf{h}_{\mathbf{k}'}\cdot\mathbf{h}_{-\mathbf{k}'})$$
$$\mathcal{F}^{\mathbf{q}\neq 0}_{\text{eff}}[\mathbf{h}] = \frac{Y_0}{8}\int (d\mathbf{k}d\mathbf{k}')\int' (d\mathbf{q})[\mathbf{k}\times\hat{\mathbf{q}}]^2[\mathbf{k}'\times\hat{\mathbf{q}}]^2(\mathbf{h}_{\mathbf{k}}\cdot\mathbf{h}_{-\mathbf{k}-\mathbf{q}})(\mathbf{h}_{\mathbf{k}'}\cdot\mathbf{h}_{-\mathbf{k}'+\mathbf{q}})$$

As is evident from Subsection 2.1.4 all information about Hooke's law is contained in the correlation function (I drop brackets and bar sign from now on)

$$K_\alpha = \int \frac{d^D\mathbf{x}}{L^D}\left\langle(\partial_\alpha\mathbf{h}\cdot\partial_{(\alpha)}\mathbf{h})\right\rangle = \int \frac{(d\mathbf{p})}{L^D}p^2_{(\alpha)}\left\langle(\mathbf{h}_{\mathbf{p}}\cdot\mathbf{h}^*_{\mathbf{p}})\right\rangle = d_c\int (d\mathbf{p})p^2_{(\alpha)}\mathcal{G}_{\mathbf{p}},$$

where $\mathcal{G}_{\mathbf{p}}$ is the exact Green's function. Before developing a perturbation theory I take a deeper look at the interaction structure.

3.1 Diagrammatics

Bare Green's function is given by

$$\left\langle h^\mu_{\mathbf{p}}h^\nu_{-\mathbf{q}}\right\rangle_0 = \int\left[\prod_{\mathbf{k},\alpha\beta}\frac{dh^\alpha_{\mathbf{k}}\wedge(dh^\beta_{\mathbf{k}})^*}{2\pi i(TL^D)^{d_c}}\right]h^\mu_{\mathbf{p}}(h^\nu_{\mathbf{q}})^*\exp\left[-\frac{1}{2TL^D}\sum_{\mathbf{p}}\left[\varkappa p^4+\sigma^0_\beta p^2_{(\beta)}\right](\mathbf{h}_{\mathbf{p}}\cdot\mathbf{h}^*_{\mathbf{p}})\right]$$
$$= (2\pi)^D\delta(\mathbf{p}-\mathbf{q})\delta^{\mu\nu}G^0_{\mathbf{p}}, \qquad G^0_{\mathbf{p}} = \frac{T}{\varkappa p^4+\sigma^0_1 p^2_1+\sigma^0_2 p^2_2}.$$

I have two distinct types of interaction presented on Figure 5.

Because of the momentum constraint, tadpole–like diagrams are only allowed for momentum non–bearing interactions. That is why there is only a self–energy-like contribution in first order

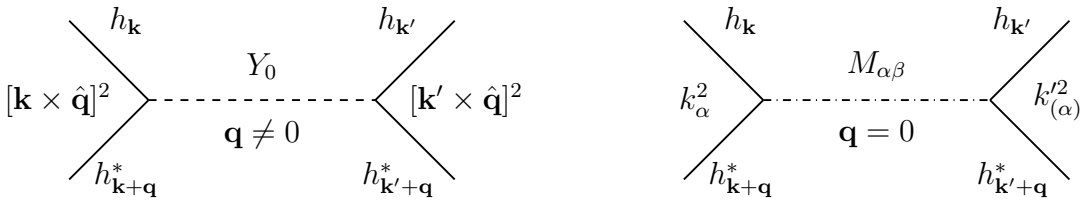


Figure 5: Interaction diagrams.

in Y_0.

$$\begin{aligned}\mathcal{G}_{\mathbf{p}}+ &= -\frac{Y_0}{T}\int (d\mathbf{k}d\mathbf{k}'d'\mathbf{q})[\mathbf{k}\times\hat{\mathbf{q}}]^2[\mathbf{k}'\times\hat{\mathbf{q}}]^2\left\langle (\mathbf{h}_{\mathbf{p}}\cdot\mathbf{h}^*_{\mathbf{p}})(\mathbf{h}_{\mathbf{k}}\cdot\mathbf{h}^*_{\mathbf{k}+\mathbf{q}})(\mathbf{h}_{\mathbf{k}'}\cdot\mathbf{h}^*_{\mathbf{k}'-\mathbf{q}})\right\rangle_0 \\ &= -\frac{d_c Y_0}{T}\int' (d\mathbf{q})[\mathbf{p}\times\hat{\mathbf{q}}]^4(2\pi)^D\delta(\mathbf{k}=0)[G^0_{\mathbf{p}}]^2 G^0_{\mathbf{p}-\mathbf{q}} \\ &= -\frac{d_c Y_0}{T}[G^0_{\mathbf{p}}]^2\int' (d\mathbf{q})[\mathbf{p}\times\hat{\mathbf{q}}]^4 G^0_{\mathbf{p}-\mathbf{q}}.\end{aligned}$$

Another correction, of the same order in L^{-D}, comes from $\mathcal{F}^{\mathbf{q}=0}_{\text{eff}}$ term with $M_{\alpha\beta}$ and the following coupling.

$$\begin{aligned}\mathcal{G}_{\mathbf{p}}+ &= -\frac{1}{T}\int \frac{(d\mathbf{k}d\mathbf{k}')}{L^D}k^2_{(\alpha)}M_{\alpha\beta}(k'_{(\beta)})^2\left\langle (\mathbf{h}_{\mathbf{p}}\cdot\mathbf{h}^*_{\mathbf{p}})(\mathbf{h}_{\mathbf{k}}\cdot\mathbf{h}^*_{\mathbf{k}})(\mathbf{h}_{\mathbf{k}'}\cdot\mathbf{h}^*_{\mathbf{k}'})\right\rangle \\ &= -\frac{d_c^2}{T}p^2_{(\alpha)}M_{\alpha\beta}\int\frac{(d\mathbf{k}')}{L^D}(k'_{(\beta)})^2(2\pi)^D\delta(\mathbf{k}=0)(2\pi)^D\delta(\mathbf{k}'=0)[G^0_{\mathbf{p}}]^2 G^0_{\mathbf{k}'} \\ &= -\frac{d_c^2}{T}p^2_{(\alpha)}M_{\alpha\beta}[G^0_{\mathbf{p}}]^2\int(d\mathbf{k}')(k'_{(\beta)})^2 G^0_{\mathbf{k}'}.\end{aligned}$$

Tadpole diagram with $2\mu k_1 k_2 k'_1 k'_2$ is zero because of its tensor structure. Diagrams corresponding to these two contributions are depicted on Figure 6.

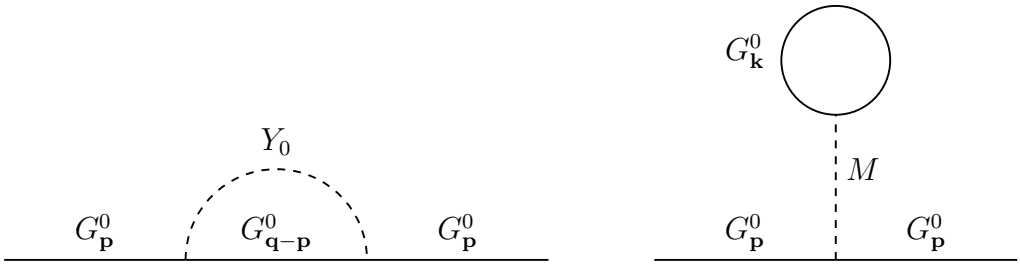


Figure 6: First–oder corrections (in interaction strength).

The problem with the former expressions is that σ^0_1, σ^0_2 are some unknown numbers here; they are connected with the true tension $\sigma_{1,2}$ through (2.11). In other words, I would like to

operate with

$$G_{\mathbf{p}} = \frac{T}{\varkappa p^4 + \sigma_1 p_1^2 + \sigma_2 p_2^2}$$

instead of $G^0_{\mathbf{p}}$. It so happens that's exactly what the momentum–free interaction $\mathcal{F}_{\mathbf{q}=0}$ is responsible for. One way to see that is to use Ward's identity [7], but a simpler way is to isolate all contributions in leading order in L^{-D}.

Infinite size L limit

Any diagram that does not vanish in the $L \to +\infty$ limit has the form of the diagram that one can produce using only the momentum–bearing interaction Y_0 with the bare Green function $G^0_{\mathbf{p}}$ replaced by $G_{\mathbf{p}}$. The function $G_{\mathbf{p}}$ is given by the series re–summed in Figure 7. It so happens

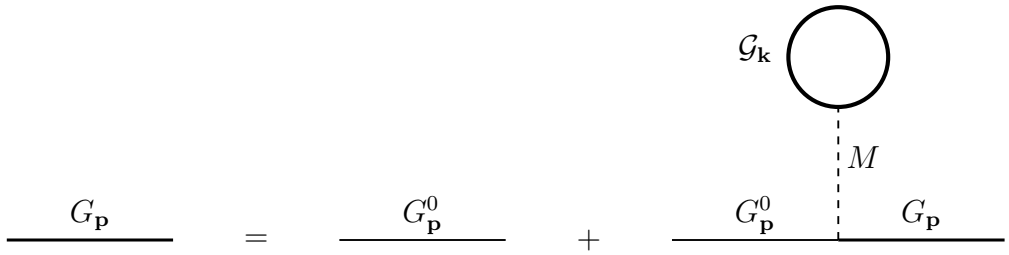


Figure 7: Stress renormalization equation.

that renormalization of σ provided by $M_{\alpha\beta}$ changes $\sigma^0 \mapsto \sigma$ exactly. Diagrammatic equation in Figure 7 reads

$$G_{\mathbf{p}} = G^0_{\mathbf{p}} - G^0_{\mathbf{p}} G_{\mathbf{p}} p^2_{(\alpha)} \frac{M_{\alpha\beta}}{2T} \int (d\mathbf{k}) k^2_{(\beta)} d_c \mathcal{G}_{\mathbf{k}}.$$

and results in correction of σ, namely,

$$G_{\mathbf{p}} = \left((G^0_{\mathbf{p}})^{-1} + \delta\sigma_\alpha p^2_{(\alpha)} \right)^{-1}, \qquad \delta\sigma_\alpha = \frac{M_{\alpha\beta}}{2} \int (d\mathbf{k}) k^2_{(\beta)} d_c \mathcal{G}_{\mathbf{k}} = \frac{1}{2} M_{\alpha\beta} \left\langle \bar{K}_\beta \right\rangle .$$

That being said, I exclude the interaction with $\mathbf{q} = 0$ completely. In other words, from now on I work with the following expression for the free energy.

$$\mathcal{F}^{(2)}_{\text{eff}}[\mathbf{h}] = \frac{1}{2} \int (d\mathbf{p}) \left[\varkappa p^4 + \sigma_\beta p^2_{(\beta)} \right] (\mathbf{h}_{\mathbf{p}} \cdot \mathbf{h}_{-\mathbf{p}}), \tag{3.1}$$

$$\mathcal{F}_{\text{eff}}[\mathbf{h}] = \frac{Y_0}{8} \int (d\mathbf{k} d\mathbf{k}') \int_{\mathbf{q}\neq 0} (d\mathbf{q}) [\mathbf{k} \times \hat{\mathbf{q}}]^2 [\mathbf{k}' \times \hat{\mathbf{q}}]^2 (\mathbf{h}_{\mathbf{k}} \cdot \mathbf{h}_{-\mathbf{k}-\mathbf{q}}) (\mathbf{h}_{\mathbf{k}'} \cdot \mathbf{h}_{-\mathbf{k}'+\mathbf{q}})$$

Green function from now on stands for

$$G_{\mathbf{p}} = \frac{T}{\varkappa p^4 + \sigma_1 p_1^2 + \sigma_2 p_2^2}.$$

and the only Y–interaction vertex present in Figure 5 is left.

High d_c expansion

Now I develop perturbation theory. Since the initial action does not contain a small parameter in the form of an anharmonic term I construct it myself by treating $\mathbf{h}$ as a vector of size d_c. Each Green function bubble is proportional to d_c, so I have to encounter the screening of $\mathcal{F}_{\text{int}}$ via the polarization bubble.

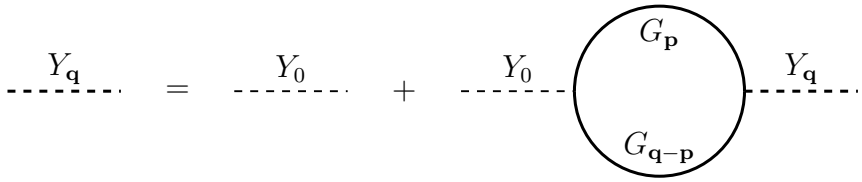


Figure 8: Screening of interaction.

Interaction constant effectively starts bearing momentum $Y_{\mathbf{q}}$ and is given by

$$\frac{Y_{\mathbf{q}}}{2} = \frac{Y_0}{2}\left(1 + \frac{Y_0}{2}\Pi_{\mathbf{q}}\right)^{-1}$$

where $\Pi_{\mathbf{q}}$ is the polarization operator given by

$$\Pi_{\mathbf{q}} = \frac{d_c}{T}\int (d\mathbf{k}) G_{\mathbf{k}} G_{\mathbf{q}-\mathbf{k}} [\mathbf{k}\times\hat{\mathbf{q}}]^4 = \frac{d_c T}{\varkappa\sigma} P\left(\frac{\mathbf{q}}{q_0}\right)$$
$$P(\mathbf{q}) = \int \frac{(d\mathbf{k})}{k^4 + k_1^2} \frac{[\mathbf{k}\times\hat{\mathbf{q}}]^4}{(\mathbf{k}-\mathbf{q})^4 + (k_1 - q_1)^2}.$$

Here $q_0 = \sqrt{\sigma/\varkappa}$ is the characteristic scale set by stress and $P(\mathbf{q})$ is the dimensionless polarization operator that is calculated in Appendix A.2. At the moment it is only important how P behaves at large argument values.

$$P(q \gg 1) = \int \frac{(d\mathbf{k})[\mathbf{k}\times\hat{\mathbf{q}}]^4}{k^4(\mathbf{k}-\mathbf{q})^4} = \frac{3}{16\pi}\frac{1}{q^2}.$$

From here a new characteristic scale set by the interaction strength q_* appears, that tends to infinity together with d_c. Since all the anomalous physics comes from the $q \ll q_*$ range, such an artificial parameter is expected to give a good approximation.

$$\frac{Y_{\mathbf{q}}}{2} \approx \frac{1}{\Pi_{\mathbf{q}}} = \frac{\varkappa\sigma}{d_c T}\frac{1}{P(\mathbf{q}/q_0)}, \qquad \frac{\sigma}{\varkappa} \equiv q_0^2 \ll q^2 \ll q_*^2 \equiv \frac{3}{16\pi}\frac{d_c Y_0 T}{\varkappa^2},$$

After the screening is accounted for, $Y_{\mathbf{q}}$ becomes explicitly small in $1/d_c \to 0$. To draw all the diagrams in n–th order one has to count the number of loops (closed Green's function lines) — each gives d_c — and the number of interaction lines — each gives $1/d_c$.

3.2 Scaling Ansatz

Here I for a moment forget about perturbation theory and describe a general statement that is known about the exact Green function. Formally, the exact Green function is given by a sum of the asymptotic perturbation series, which can be formally rewritten as the series for the exact self–energy $\mathbf{\Sigma_p}$ and exact polarization operator $\mathbf{\Pi_q}$.

$$\mathcal{G}_\mathbf{p} = \left(G_\mathbf{p}^{-1} + T\mathbf{\Sigma_p}\right)^{-1} \qquad \frac{\mathcal{Y}_\mathbf{q}}{2} = \frac{Y_0}{2}\left(1 + \mathbf{\Pi_q}\frac{Y_0}{2}\right)^{-1}. \tag{3.2}$$

In the diagrams below the solid line denotes the exact Green function $\mathcal{G}_\mathbf{p}$ and the dashed line stands for $\mathcal{Y}_\mathbf{q}$ — the interaction screened with the exact polarization operator. There is an

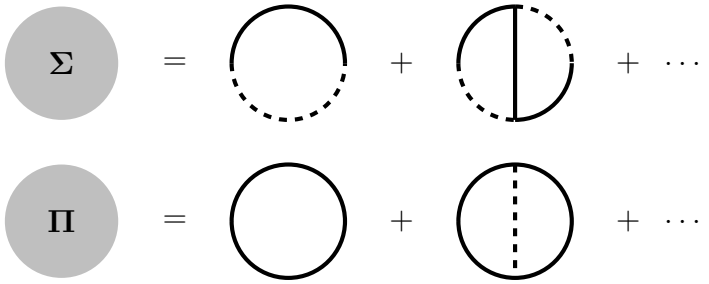


Figure 9: Formal asymptotic series for self–energy and polarization operator.

ansatz for the exact Green function

$$\mathcal{G}_\mathbf{p} = \frac{T}{\varkappa_\mathbf{p} p^4 + \sigma_1 p_1^2 + \sigma_2 p_2^2}, \qquad \varkappa_\mathbf{p} = \varkappa \begin{cases} 1, & p \gg p_*, \\ (p_*/p)^\eta, & p \ll p_*. \end{cases} \tag{3.3}$$

That asymptotically satisfies equations (3.2) since the scaling of the bending rigidity implies the same scaling for any diagram in the series presented in Figure 9.

$$\Sigma_\mathbf{p} \propto \frac{\varkappa}{d_c T}\left(\frac{p_*}{p}\right)^\eta p^4, \quad \Pi_\mathbf{q} \propto \frac{d_c T}{\varkappa^2 q^2}\left(\frac{q}{p_*}\right)^{2\eta}, \qquad p, q \ll p_* = \sqrt{\frac{3A}{16\pi}\frac{d_c Y_0 T}{\varkappa^2}}.$$

Here p_* coincides with the scale set by the interaction strength defined in the previous section up to an unknown prefactor A of order unity and $\eta > 0$ is an unknown critical index. The number η is unambiguously connected with the exponent (1.1) from anomalous Hooke's law

$$\alpha = \frac{\eta}{2 - \eta}$$

as will be shown later, so it is of great interest to find that number.

3.2.1 Self–consistent screening approximation

Self–consistent screening approximation (SCSA) allows [9] one to find η in some manner; it has no controllable parameter, yet is believed to give numerically satisfactory results [5]. The idea of the approximation is to take only the first terms in the series in Figure 9, i e. solve the following system of equations.

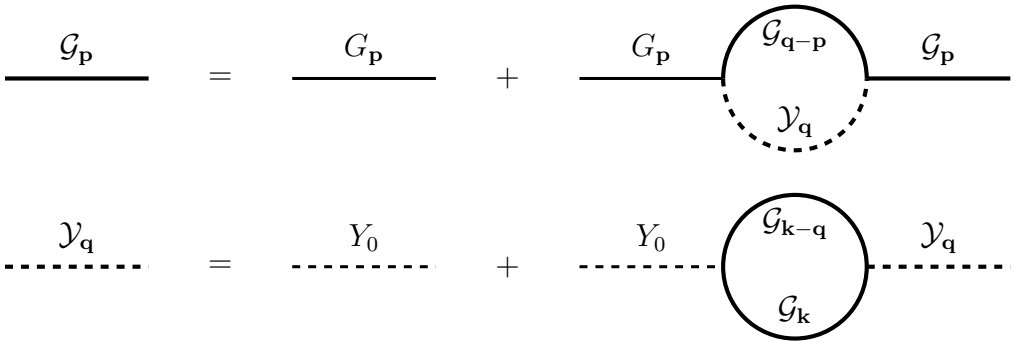


Figure 10: System of self–consistent screening approximation equations.

These diagrams (Figure 10) correspond to analytical expressions

$$\Pi_{\mathbf{q}} = \frac{d_c}{T}\int (d\mathbf{k})\mathcal{G}_{\mathbf{k}}\mathcal{G}_{\mathbf{k-q}}[\mathbf{k}\times\hat{\mathbf{q}}]^4, \qquad \mathcal{G}_{\mathbf{p}} = \left(G_{\mathbf{p}}^{-1}+\Sigma_{\mathbf{p}}\right)^{-1}$$

$$T\Sigma_{\mathbf{p}} = \int' (d\mathbf{q})\mathcal{G}_{\mathbf{p-q}}Y_{\mathbf{q}}[\mathbf{p}\times\hat{\mathbf{q}}]^4, \qquad \frac{Y_{\mathbf{q}}}{2} = \frac{Y_0}{2}\left(1+\Pi_{\mathbf{q}}\frac{Y_0}{2}\right)^{-1}.$$

In the absence of strain ($\sigma = 0$) there are two regimes presented in (3.3). For $p \gg p_*$, I assume the integral over $\mathbf{q}$ comes from $q \sim p$ and substitute $Y_{\mathbf{q}} \sim Y_0$ and see my assumption satisfied. For $p \ll p_*$, I guess power law dependence $\varkappa_{\mathbf{p}} = \varkappa(Ap_*/p)^\eta$.

$$\Pi_{\mathbf{q}} = A^{2\eta}\frac{d_c T}{\varkappa^2 p_*^{2\eta}}\Pi(\eta,\eta)q^{2\eta-2}, \qquad \Pi(\eta,\eta') = \int \frac{[\mathbf{k}\times\hat{\mathbf{q}}]^4(d\mathbf{k})}{k^{4-\eta}|\mathbf{k}-\hat{\mathbf{q}}|^{4-\eta'}}.$$

$$\Sigma_{\mathbf{p}} = \frac{2A^\eta\Sigma(\eta,\eta)}{Td_c\Pi(\eta,\eta)}\varkappa\left(\frac{p_*}{p}\right)^\eta p^4, \qquad \Sigma(\eta,\eta') = \int' \frac{[\hat{\mathbf{p}}\times\hat{\mathbf{q}}]^4 q^{2-2\eta}}{(\hat{\mathbf{p}}-\mathbf{q})^{4-\eta'}}(d\mathbf{q}).$$

Functions $\Sigma(\eta,\eta)$ and $\Pi(\eta,\eta)$ are calculated, for example, in [13, 5]. Self–consistency demands

$$\frac{2\Sigma(\eta,\eta)}{d_c\Pi(\eta,\eta)} = 1 \qquad \Rightarrow \qquad \eta = \frac{4}{d_c+\sqrt{16-2d_c+d_c^2}}\bigg|_{d_c=1} \simeq 0.82 \qquad \eta \sim \frac{2}{d_c},\ d_c\to\infty.$$

Number A may not be determined from that consideration.

The problem with that result, however, is that it is not done under any controllable approximation. That's why I am not satisfied with it and develop a perturbation theory in $1/d_c$.

3.3 Critical exponent expansion

In order to find the critical exponent η as a function of the complementary dimensionality d_c, I calculate the small-argument asymptotics of the self–energy (without external stress $\sigma = 0$) perturbatively in $1/d_c$. The scaling ansatz (SA) tells me that at small momenta the bending rigidity behaves as

$$\frac{\varkappa_{\mathbf{p}}}{\varkappa} = 1 - \frac{T\Sigma_{\mathbf{p}}}{\varkappa p^4} \sim \left[A\frac{p_*}{p}\right]^{\eta}, \qquad p \ll p_*, \tag{3.4}$$

where the exponent $\eta = \eta(d_c)$ is the number I seek to calculate up to the second order in $1/d_c$.

Averaging rules Throughout the calculations I am going to frequently use the following identities

$$\left\langle \frac{[\mathbf{p}\times\mathbf{q}]^4}{(\mathbf{p}-\mathbf{q})^4} \right\rangle_{\hat{\mathbf{q}}} = \frac{3}{8}\min\{p^4, q^4\}, \tag{3.5}$$

$$\left\langle \frac{[\mathbf{p}\times\mathbf{q}]^4}{(\mathbf{p}-\mathbf{q})^2} \right\rangle_{\hat{\mathbf{q}}} = \min\{p^4, q^4\}\left(\max\{p^2, q^2\} - \frac{1}{3}\min\{p^2, q^2\}\right), \tag{3.6}$$

$$\left\langle \frac{[\mathbf{p}\times\mathbf{q}]^2}{(\mathbf{p}-\mathbf{q})^4} \right\rangle_{\hat{\mathbf{q}}} = \frac{1}{2\max\{p^2, q^2\}|p^2 - q^2|}, \tag{3.7}$$

3.3.1 First–order self–energy

Expansion of self–energy in $1/d_c$ starts from the trivial term (Figure 11–11*a*)

$$T\Sigma^{(1)}_{\mathbf{p}} = -\frac{Y_0 T}{\varkappa}\int \frac{[\mathbf{p}\times\hat{\mathbf{q}}]^4 (d\mathbf{q})}{(\mathbf{p}-\mathbf{q})^4(1+\frac{Y_0}{2}\Pi_{\mathbf{q}})} = -\frac{\varkappa p_*^4}{2d_c}\mathcal{S}\left(\sqrt{2}\frac{p}{p_*}\right).$$

In the absence of stress $\sigma = 0$ the polarization operator is simply

$$Y_0\Pi_{\mathbf{q}} = \frac{q_*^2}{q^2}, \qquad q_*^2 = \frac{3}{16\pi}\frac{d_c Y_0 T}{\varkappa^2},$$

and the dimensionless self–energy is given by the function

$$\begin{aligned}\mathcal{S}(\mathbf{p}) &= \frac{16\pi}{3}\int \frac{[\mathbf{p}\times\hat{\mathbf{q}}]^4 (d\mathbf{q})}{(\mathbf{p}-\mathbf{q})^4 + (\mathbf{p}-\mathbf{q})^2} = \frac{16\pi}{3}\int \frac{[\mathbf{p}\times\mathbf{q}]^4 (d\mathbf{q})}{(\mathbf{p}-\mathbf{q})^4(q^4+q^2)} = (3.5) = \\ &= \frac{1}{2}\int_0^\infty \frac{dq}{q^2+q}\min\{p^4, q^2\} = \frac{1}{2}\left\{(p^4-1)\ln\left(1+p^2\right) - p^4\ln p^2 + p^2\right\}.\end{aligned}$$

Therefore the first–order term has the following asymptotics

$$T\Sigma^{(1)}_{\mathbf{p}} \sim \frac{2}{d_c}\varkappa p^4 \ln\left[\frac{\sqrt{2}}{e^{1/4}}\frac{p}{p_*}\right], \quad p \ll p_*$$

That allows me to find

$$A(d_c) = \frac{e^{1/4}}{\sqrt{2}} + \frac{A_1}{d_c} + \ldots, \quad \eta(d_c) = \frac{2}{d_c} + \frac{\eta_2}{d_c^2} + \ldots \tag{3.8}$$

3.3.2 Second order. SCSA–like contributions

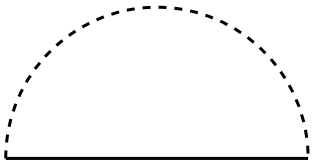
Second–order consists of two trivial SCSA–like contributions and four non–SCSA corrections. Together these two give the result that may be found from

$$\eta_{\text{SCSA}}(d_c) = \frac{4}{d_c + \sqrt{16 - 2d_c + d_c^2}} = \frac{2}{d_c} + \frac{1}{d_c^2} + \mathcal{O}\left(\frac{1}{d_c^3}\right).$$

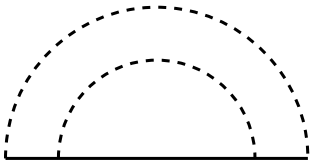
Plugging (3.8) into (3.4) with $\eta_2 = 1 + \delta\eta_2$ and expanding in $1/d_c$ I come to

$$d_c^2 \frac{T\Sigma^{(2)}_{\mathbf{p}}}{\varkappa p^4} = -2\ln^2\left[\sqrt{2}\frac{p}{p_*}\right] + (2 + \delta\eta_2)\ln\left[\sqrt{2}\frac{p}{p_*}\right] - \frac{3}{8} - \frac{\delta\eta_2}{4} - 2A_1. \tag{3.9}$$

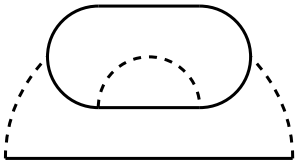
I see that with logarithmic accuracy SCSA–like contributions to self–energy should produce at small momenta $-2\ln^2[\sqrt{2}p/p_*] + 2\ln[p/p_*]$. Let me check that.



(11a) Fan diagram.



(12a) Rainbow diagram



(12b) Upset diagram

Figure 11: SCSA–like self–energy corrections up to the second order.

Rainbow diagram Explicit expression for the diagram in Figure 11–12a is

$$T\Sigma^{(2a)}_{\mathbf{p}} = \frac{\varkappa p_*^4}{d_c^2}\mathcal{S}^{(2a)}\left(\sqrt{2}\frac{p}{p_*}\right), \qquad \mathcal{S}^{(2a)}(p) = \left(\frac{16\pi}{3}\right)\int \frac{(d\mathbf{q})[\mathbf{p}\times\mathbf{q}]^4 q^{-8}\mathcal{S}(q)}{(\mathbf{p}-\mathbf{q})^4 + (\mathbf{p}-\mathbf{q})^2}$$

Since I only need the asymptotic at small p, I calculate

$$\begin{aligned}\tilde{\mathcal{S}}^{(2a)}(p) &= \left(\frac{16\pi}{3}\right)\int (d\mathbf{q})\frac{[\mathbf{p}\times\mathbf{q}]^4\mathcal{S}(q)}{(\mathbf{p}-\mathbf{q})^2q^8} = (3.6) = \\ &= \frac{1}{2}\int_0^\infty \frac{dq}{q^4}\mathcal{S}\left(\sqrt{q}\right)\min\{p^4,q^2\}\left(\max\{p^2,q\}-\frac{1}{3}\min\{p^2,q\}\right) \\ &\sim \frac{1}{2}\left(\ln^2 p - \ln p + \frac{21+2\pi^2}{24}\right)p^4.\end{aligned}$$

since it is convergent, it is enough to find the leading asymptotic; however, it happens to be not enough to find the number under the logarithm. What was left is the following integral

$$\mathcal{S}^{(2a)}(p) - \tilde{\mathcal{S}}^{(2a)}(p) = -\left(\frac{16\pi}{3}\right)\int \frac{(d\mathbf{q})[\mathbf{p}\times\mathbf{q}]^4\mathcal{S}(q)}{((\mathbf{p}-\mathbf{q})^2+1)\,q^8} \sim \frac{-1}{2}p^4\int_0^\infty \frac{dq}{q^2}\frac{\mathcal{S}\left(\sqrt{q}\right)}{1+q} = \left(3-\pi^2\right)\frac{p^4}{12}.$$

As a result,

$$\frac{T\Sigma_{\mathbf{p}}^{(2a)}}{\varkappa p^4} \sim \frac{2}{d_c^2}\left(\ln^2\left[\sqrt{2}\frac{p}{p_*}\right] - \ln\left[\sqrt{2}\frac{p}{p_*}\right] + \frac{33-2\pi^2}{24}\right).$$

Upset diagram Expression for such a diagram is

$$\begin{aligned}T\Sigma_{\mathbf{p}}^{(2b)} &= d_c\int (d\mathbf{q}d\mathbf{k})\left(\frac{Y_{\mathbf{q}}}{T}\right)^2 G_{\mathbf{p}-\mathbf{q}}G_{\mathbf{k}-\mathbf{q}}G_{\mathbf{k}}^2 T\Sigma_{\mathbf{k}}^{(1)}[\mathbf{p}\times\hat{\mathbf{q}}]^4[\mathbf{k}\times\hat{\mathbf{q}}]^4 \\ &= -\frac{2}{d_c^2}\varkappa p_*^4\mathcal{S}^{(2b)}\left(\sqrt{2}\frac{p}{p_*}\right), \\ \mathcal{S}^{(2b)}(p) &= \left(\frac{16\pi}{3}\right)^2\int\frac{(d\mathbf{q})[\mathbf{p}\times\mathbf{q}]^4}{(q^4+q^2)(\mathbf{p}-\mathbf{q})^4}\int\frac{(d\mathbf{k})}{k^8}\frac{[\mathbf{k}\times\mathbf{q}]^4}{(\mathbf{k}-\mathbf{q})^4}\mathcal{S}(k) = (3.5) = \\ &= \frac{1}{4}\int_0^\infty\frac{\min\{q^2,p^4\}}{(q^2+q)^2}\int_0^\infty\frac{dk}{k^4}\min\{k^2,q^2\}\mathcal{S}\left(\sqrt{k}\right) \\ &\sim \frac{p^4}{2}\left(\ln^2 p - \ln p + \frac{3}{8} - \frac{\pi^2}{12}\right).\end{aligned}$$

As a result,

$$\frac{T\Sigma_{\mathbf{p}}^{(2b)}}{\varkappa p^4} \sim \frac{4}{d_c^2}\left(-\ln^2\left[\sqrt{2}\frac{p}{p_*}\right] + \ln\left[\sqrt{2}\frac{p}{p_*}\right] - \frac{3}{8} + \frac{\pi^2}{12}\right).$$

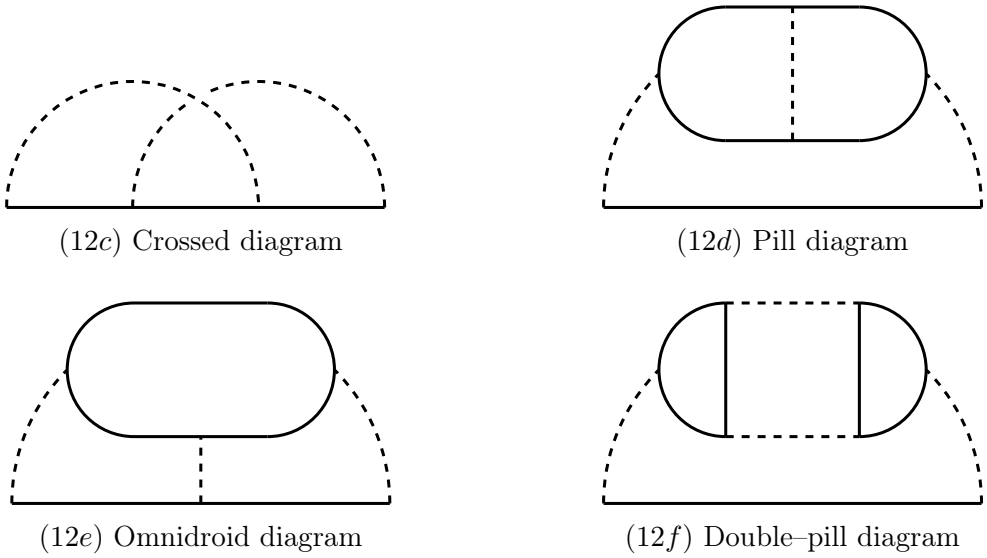
Sum of SCSA–like diagrams Two SCSA–like second order corrections together give

$$d_c^2\frac{T\Sigma_{\mathbf{p}}^{(2ab)}}{\varkappa p^4} = -2\ln^2\left[\sqrt{2}\frac{p}{p_*}\right] + 2\ln\left[\sqrt{2}\frac{p}{p_*}\right] + \frac{15+2\pi^2}{12}.$$

So good so far. Next, I'm interested in non–SCSA corrections.

3.3.3 Second order. Non–SCSA contributions

In order to find $\delta\eta_2$ I need to calculate the following four diagrams.



(12c) Crossed diagram

(12d) Pill diagram

(12e) Omnidroid diagram

(12f) Double–pill diagram

Figure 12: Non–SCSA contributions

Here I provide the summary of the results and details of the calculation are presented in the next paragraphs. Asymptotics of the diagrams in Figure 12 at $p \ll p_*$ are as follows.

$$d_c^2 \frac{T\Sigma_{\mathbf{p}}^{(2c)}}{\varkappa p^4} = -\frac{7}{3}\ln\left[\frac{p}{p_*}\right] + \text{const}, \tag{3.10}$$

$$d_c^2 \frac{T\Sigma_{\mathbf{p}}^{(2d)}}{\varkappa p^4} = +2\ln\left[\frac{p}{p_*}\right] + \text{const}, \tag{3.11}$$

$$d_c^2 \frac{T\Sigma_{\mathbf{p}}^{(2e)}}{\varkappa p^4} = +\frac{58}{27}\ln\left[\frac{p}{p_*}\right] + \text{const}, \tag{3.12}$$

$$d_c^2 \frac{T\Sigma_{\mathbf{p}}^{(2f)}}{\varkappa p^4} = -\frac{3+68\zeta(3)}{27}\ln\left[\frac{p}{p_*}\right] + \text{const}. \tag{3.13}$$

Thus, according to (3.9), the difference between SCSA and the true value of the critical exponent is seen in $1/d_c^2$–order and equals

$$\delta\eta_2 = \lim_{d_c\to\infty} \frac{\eta - \eta_{\text{SCSA}}}{d_c^{-2}} = -\frac{7}{3} + 2 + \frac{58}{27} - \frac{3+68\zeta(3)}{27} = \frac{46-68\zeta(3)}{27} \simeq -1.32.$$

Critical exponent from Hooke's law is then $\alpha = \alpha_{\text{SCSA}} + \delta\alpha_2/d_c^2$ with $\delta\alpha_2 = 2(\alpha_{\text{SCSA}}/\eta_{\text{SCSA}})^2\delta\eta_2$.

$$\frac{1}{1+2\alpha} = \frac{1}{1+2\alpha_{\text{SCSA}}}\left(1 - \frac{33-2\sqrt{15}}{21}\frac{\delta\eta_2}{d_c^2} + \dots\right) \simeq \frac{1}{1+2\alpha_{\text{SCSA}}}\left(1 + \frac{1.6}{d_c^2} + \dots\right).$$

3.4 Absolute Poisson ratio

I would like to find the absolute Poisson ratio in the regime $\sigma \ll \sigma_*$ of small stress.

$$\nu = \frac{\nu_0 - Y_0 \delta K_2/\sigma}{1 + Y_0 \delta K_1/\sigma} \approx -\frac{\delta K_2}{\delta K_1}.$$

I would like to construct an expansion in $1/d_c$ using the perturbation theory in the screened interaction developed in the previous section. However, I notice that the zeroth order phonon correlator diverges at small momenta

$$\delta K_\beta^{(0)} = K_\beta^{(0)}(0) - K_\beta^{(0)}(\sigma) = d_c T \int \frac{(d\mathbf{p}) p_{(\beta)}^2 \sigma p_1^2}{\varkappa p^4 (\varkappa p^4 + \sigma p_1^2)}.$$

To see that explicitly I rewrite integrals as follows.

$$\begin{aligned}
\delta K_1^{(0)} - \delta K_2^{(0)} &= \frac{\ln 2}{4\pi} \frac{d_c T}{\varkappa}, \\
\delta K_1^{(0)} + \delta K_2^{(0)} &= 2 \left[\int \frac{\frac{\sigma}{\varkappa}(d\mathbf{p})}{p^2 (p^2 + \frac{\sigma}{\varkappa})} - \frac{\ln 2}{4\pi} \right] \frac{d_c T}{\varkappa}.
\end{aligned}$$

Exact correlator does not have such a problem since the self–energy removes the divergence.

$$\delta K_\beta = d_c T \int \left[\frac{1}{\varkappa p^4 - T\Sigma_p^{\sigma=0}} - \frac{1}{\varkappa p^4 + \sigma p_1^2 - T\Sigma_{\mathbf{p}}} \right] p_\beta^2 (d\mathbf{p})$$

Scaling Ansatz (SA) tells me that in the absence of stress $\sigma = 0$ the renormalized bending rigidity behaves as

$$\varkappa_{\mathbf{p}} = \varkappa - p^{-4} T \Sigma_p^{\sigma=0} \sim \varkappa \left(A \frac{p_*}{p} \right)^\eta, \qquad p^2 \ll p_*^2 = \frac{3}{16\pi} \frac{d_c Y_0 T}{\varkappa^2},$$

where the critical exponent η is believed to be close to 0.8 and the number A is of order unity, which will be hidden in the redefinition of p_* till the end of that section.

It is clear that in the case of finite stress such behavior is only valid when the characteristic momenta are large enough to disregard the σp^2 term, i.e. for $p \gg p_\sigma = p_0 p_*^{-\alpha}$, $\alpha = \eta/(2-\eta)$. There are some assumptions about how the rigidity renormalizes at finite strain [16]; however, knowing the asymptotic behavior of $\varkappa_{\mathbf{p}}$ at $p \ll p_\sigma$ is not enough to calculate the Poisson ratio since the integral for the correlation function δK comes from all momenta $p \ll p_*$ as may be seen from the following model.

3.4.1 Oversimplified model

Here I calculate the correlational function in the framework of the model

$$\varkappa^\sigma_{\mathbf{p}} = \varkappa \begin{cases} 1, & p_\sigma < p_* < p \\ (p_*/p)^\eta, & p_\sigma < p < p_* \\ (p_*/p_\sigma)^\eta, & p < p_\sigma < p_* \end{cases} \qquad p_\sigma = x p_0 \left(\frac{p_0}{p_*}\right)^\alpha = x p_* \left(\frac{p_0}{p_*}\right)^{1+\alpha} \ll p_*$$

I keep the number $x \sim 1$ in order to see what part of the answer is going to be cut-off-dependent. In what follows p_* may be considered the ultraviolet cut-off since contributions coming from higher momenta have weaker scaling with stress. For $\sigma \ll \sigma_*$

$$\begin{aligned} \frac{\varkappa}{d_c T}\delta K_\beta &\sim \int \frac{\varkappa\sigma p_\beta^2 p_1^2 (d\mathbf{p})}{\varkappa_p p^4(\varkappa_p p^4 + \sigma p_1^2)} + \int_{p<p_\sigma} (d\mathbf{p}) \left[\frac{\varkappa p_\beta^2}{\varkappa^\sigma_{\mathbf{p}} p^4 + \sigma p_1^2} - \frac{\varkappa p_\beta^2}{\varkappa_{\mathbf{p}} p^4 + \sigma p_1^2}\right] \\ &= \underbrace{\frac{1+\alpha}{4\sin\pi\alpha}\left\langle \hat{p}_1^{2\alpha}\hat{p}_\beta^2\right\rangle_{\hat{\mathbf{p}}}}_{A_\beta^{[-1]}\sim d_c}\left(\frac{\sigma}{\sigma_*}\right)^\alpha + \underbrace{\int_{p<x}(d\mathbf{p})\left[\frac{p_\beta^2}{p^4+p_1^2} - \frac{p_\beta^2}{p^{4-\eta}+p_1^2}\right]}_{A_\beta^{[1]}\sim d_c^{-1}}\left(\frac{\sigma}{\sigma_*}\right)^\alpha \end{aligned}$$

In the limit $d_c \to \infty$ the first term behaves as $\alpha^{-1} \sim d_c$ and the second turns into zero together with $\eta \propto d_c^{-1}$. One could also see that the dependence on the cut-off x is only present in the second contribution. In that model the expansion of PR in inverse powers of d_c looks like

$$\begin{aligned} \nu &= -\frac{A_2^{[-1]} + A_2^{[1]} + \dots}{A_1^{[-1]} + A_1^{[1]} + \dots} \sim -\frac{\langle \hat{p}_1^{2\alpha}\hat{p}_2^2\rangle_{\hat{\mathbf{p}}}}{\langle \hat{p}_1^{2\alpha+2}\rangle_{\hat{\mathbf{p}}}}\left(1 + \frac{A_2^{[1]}}{A_2^{[-1]}} - \frac{A_1^{[1]}}{A_1^{[-1]}}\right)_{\eta\to 0} \\ &= \frac{-1}{1+2\alpha}\left(1 - \eta^2 c_x\right) + \mathcal{O}\left(\eta^3\right) \end{aligned}$$

Integrals $A_\beta^{[1]}$ strongly depend on the cut-off x.

$$\begin{aligned} c_x = \frac{4\pi}{\eta}\left(A_1^{[1]} - A_2^{[1]}\right)_{\eta\to 0} &\sim 4\pi \int_{p<x} \frac{\hat{p}_2^2 - \hat{p}_1^2}{(p^2+\hat{p}_1^2)^2} p^2 \ln p\,(d\mathbf{p}) = -\int_0^x \frac{\ln p\, dp}{(1+p^2)^{3/2}} \\ &= \operatorname{arcsh} x - \frac{x\ln x}{\sqrt{1+x^2}} \end{aligned}$$

Number c_x varies between 0 and $c_1 = \operatorname{arcsh} 1 \simeq 0.88$; as $x \to +\infty$ it tends to $\ln 2 \simeq .69$.

Of course, such an oversimplified model could not produce correct numbers since the integrals are determined by the full crossover area $p \sim p_\sigma$ and the exact $\varkappa^\sigma_{\mathbf{p}}$ becomes strongly anisotropic at $p \ll p_\sigma$. Yet that model happens to correctly predict the fact shown below that the naive answer $(1+2\alpha)^{-1}$ is valid up to the first two orders in $1/d_c$.

3.4.2 Regular perturbation theory

In order to cope with the divergence, yet be able to develop a perturbation theory in $1/d_c$, I propose to expand δK_β in $\delta\Sigma_{\mathbf{p}} = \Sigma_{\mathbf{p}}^{\sigma=0} - \Sigma_{\mathbf{p}}$.

$$\begin{aligned}\delta K_\beta &= d_c T \int \left[\frac{1}{\varkappa p^4 - T\Sigma_p^{\sigma=0}} - \frac{1}{\varkappa p^4 + \sigma p_1^2 - T\Sigma_p^{\sigma=0}} + \frac{\delta\Sigma_{\mathbf{p}}}{(\varkappa p^4 + \sigma p_1^2 - T\Sigma_p^{\sigma=0})^2} + \dots \right] p_\beta^2 (d\mathbf{p}) \\ &= \underbrace{d_c T \int \frac{\sigma p_\beta^2 p_1^2 (d\mathbf{p})}{\varkappa_p p^4 (\varkappa_p p^4 + \sigma p_1^2)}}_{\delta K_\beta^{[0]} \sim d_c^2} + \underbrace{d_c T \int \frac{p_\beta^2 (d\mathbf{p})}{(\varkappa_p p^4 + \sigma p_1^2)^2} T\delta\Sigma_{\mathbf{p}}^{(1)}}_{\delta K_\beta^{[1]} \sim d_c^0} + \underbrace{d_c T \int (d\mathbf{p}) p_\beta^2 \dots}_{\delta K_\beta^{[2]} \sim d_c^{-1}} \end{aligned} \tag{3.14}$$

Here I note that in the last line I defined $\delta K_\beta^{[1]}$ with $\delta\Sigma_{\mathbf{p}} = \delta\Sigma_{\mathbf{p}}^{(1)}$. Hence $\delta K_\beta^{[2]}$ contains contributions both like $[\delta\Sigma_{\mathbf{p}}^{(1)}]^2$ and like $\delta\Sigma_{\mathbf{p}}^{(2)}$.

Since $\Sigma_{\mathbf{p}}$ has a clear expansion in $1/d_c$, expression (3.14) allows me to find δK_β up to the desired order in $1/d_c$. The problem is that I actually have two small parameters now and, strictly speaking, I need to take the limit $\sigma/\sigma_* \to 0$ first, i.e. I have to sum up the logarithms $\ln \sigma/\sigma_*$ in all orders in $1/d_c$. Let me forget about that for a moment and take a closer look at $\delta K_\beta^{[0]}$.

Leading term is convergent and may be found in the limit $\sigma \ll \sigma_*$ in terms of η.

$$\begin{aligned}\delta K_\beta^{[0]} &\sim \frac{d_c T}{2\pi} \int_0^{p_*} \frac{dp}{p} \left\langle \frac{\sigma \hat{p}_\beta^2 \hat{p}_1^2}{\varkappa (p_*/p)^\eta (\varkappa p_*^\eta p^{2-\eta} + \sigma \hat{p}_1^2)} \right\rangle_{\hat{\mathbf{p}}} \\ &\sim \frac{d_c T}{2\pi} \int_0^\infty \frac{dp \left\langle \hat{p}_1^{2\alpha} \hat{p}_\beta^2 \right\rangle_{\hat{\mathbf{p}}}}{p^{1-\eta}(p^{2-\eta}+1)} \frac{\sigma^\alpha}{(\varkappa p_*^\eta)^{\alpha+1}}, \qquad \alpha = \frac{\eta}{2-\eta}, \quad \eta = \frac{2\alpha}{1+\alpha}, \\ &= \frac{d_c T}{\varkappa} \left(\frac{\sigma}{\sigma_*}\right)^\alpha I_\alpha \left\langle \hat{p}_1^{2\alpha} \hat{p}_\beta^2 \right\rangle_{\hat{\mathbf{p}}}, \qquad I_\alpha = \int_0^\infty \frac{p^{\eta-1} dp/2\pi}{(p^{2-\eta}+1)} = \frac{1+\alpha}{4 \sin \pi\alpha}. \end{aligned}$$

Thrown away part of the integral is of order $\sigma/\sigma_* \ll (\sigma/\sigma_*)^\alpha$.

$$\frac{Y_0 \delta K_\beta^{[0]}}{\sigma} \sim \frac{16\pi}{3} I_\alpha \left\langle \hat{p}_1^{2\alpha} \hat{p}_\beta^2 \right\rangle_{\hat{\mathbf{p}}} \left(\frac{\sigma}{\sigma_*}\right)^{\alpha-1}, \qquad \sigma \ll \sigma_*.$$

Angle averages may be expressed via the Euler Γ–function, but for the PR only their ratio is important.

$$\nu^{[0]} = -\frac{\delta K_2^{[0]}}{\delta K_1^{[0]}} = -\frac{\langle \hat{p}_1^{2\alpha} \hat{p}_2^2 \rangle_{\hat{\mathbf{p}}}}{\langle \hat{p}_1^{2\alpha+2} \rangle_{\hat{\mathbf{p}}}} = -\frac{1}{1+2\alpha} = -1 + \frac{2}{d_c} + \mathcal{O}\left(\frac{1}{d_c^2}\right).$$

Since (3.14) is an expansion in $1/d_c$, it only makes sense to keep α up to the desired order in $1/d_c$.

Hypothesis I see that both logarithmic series next to d_c^2 and d_c summed up to the same function of σ. I make a hypothesis that in the limit $\sigma \ll \sigma_*$ the dependence of δK_β on σ is the same as for $\delta K_\beta^{[0]}$ and only the number in front is renormalized.

$$\frac{\varkappa}{T}\delta K_\beta = C_\beta(d_c)\left(\frac{\sigma}{\sigma_*}\right)^\alpha, \qquad C_\beta(d_c) = d_c I_\alpha \left\langle \hat{p}_1^{2\alpha}\hat{p}_\beta^2\right\rangle_{\hat{\mathbf{p}}}\left(1+\frac{\#_\beta^{(0,0)}}{d_c^2}+\mathcal{O}\left(\frac{1}{d_c^3}\right)\right).$$

That assumption may be checked in every order of perturbation theory, where I can neglect the difference between $\Sigma_{\mathbf{p}}$ and $\Sigma_{\mathbf{p}}^{\sigma=0}$ in every Green's function since I already have $\delta\Sigma_{\mathbf{p}}$ in front. Let me show how it is done for $\delta K^{[1]}$.

$$\Pi_{\mathbf{q}}^{(\eta)} = \frac{d_c}{T}\int \frac{T^2[\mathbf{k}\times\hat{\mathbf{q}}]^4(d\mathbf{k})}{(\varkappa p_*^\eta k^{4-\eta}+\sigma k_1^2)(\varkappa p_*^\eta(\mathbf{q}-\mathbf{k})^{4-\eta}+\sigma(q_1-k_1)^2)} = \frac{d_c T}{\varkappa^2 p_\sigma^2}\left(\frac{p_\sigma}{p_*}\right)^{2\eta} P_\eta\left(\frac{\mathbf{q}}{p_\sigma}\right),$$
$$T\Sigma_{\mathbf{p}}^{(\eta)} = \int\left[\frac{2T}{\Pi_{\mathbf{q}}}\frac{[\mathbf{p}\times\hat{\mathbf{q}}]^4(d\mathbf{q})}{\varkappa p_*^\eta(\mathbf{p}-\mathbf{q})^{4-\eta}+\sigma(p_1-q_1)^2} - \frac{2T}{\Pi_{\mathbf{q}}^{\sigma=0}}\frac{[\mathbf{p}\times\hat{\mathbf{q}}]^4(d\mathbf{q})}{\varkappa p_*^\eta(\mathbf{p}-\mathbf{q})^{4-\eta}}\right] = \frac{\varkappa}{d_c}p_\sigma^4\left(\frac{p_*}{p_\sigma}\right)^\eta S_\eta\left(\frac{\mathbf{p}}{p_\sigma}\right),$$

where Π_η and S_η are some dimensionless functions.

$$\delta K_\beta^{[1]} = \frac{d_c T}{\varkappa}\left(\frac{p_\sigma}{p_*}\right)^{2\eta}\int\frac{(d\mathbf{p})p_\beta^2 S_\eta(\mathbf{p})}{(p^{4-\eta}+p_1^2)^2} = \#_\beta(\alpha)\frac{d_c T}{\varkappa}\left(\frac{\sigma}{\sigma_*}\right)^\alpha.$$

where characteristic scale is

$$p_\sigma = p_0\left(\frac{p_0}{p_*}\right)^\alpha = p_*\left(\frac{p_0}{p_*}\right)^{1+\alpha} \ll p_0 \ll p_*$$

It follows from that hypothesis that the information about the Poisson ratio is encoded in the numbers $\#_\beta^{(n,0)}$; in particular, the number $\#^{(0,0)}$ may be found from $\delta K^{[1]}$ with $\varkappa_{\mathbf{p}} = \varkappa$.

Correction to the leading order value $\nu^{[0]}$ of the Poisson ratio comes from $\delta K_\beta^{[1]}$. In the limit $d_c \to \infty$ the correlator $\delta K_\beta^{[1]}$ is simply a number

$$\delta K_\beta^{[1]} \underset{d_c\to\infty}{=} d_c T\int\frac{p_\beta^2(d\mathbf{p})}{(\varkappa p^4+\sigma p_1^2)^2}\int\left[\frac{2T}{\Pi_{\mathbf{q}}}\frac{[\mathbf{p}\times\hat{\mathbf{q}}]^4}{\varkappa(\mathbf{p}-\mathbf{q})^4+\sigma(p_1-q_1)^2} - \frac{2T}{\Pi_{\mathbf{q}}^{\sigma=0}}\frac{[\mathbf{p}\times\hat{\mathbf{q}}]^4}{\varkappa(\mathbf{p}-\mathbf{q})^4}\right](d\mathbf{q})$$
$$= \frac{2T}{\varkappa}\int\frac{p_\beta^2(d\mathbf{p})}{(p^4+p_1^2)^2}\int\left[\frac{1}{P(\mathbf{q})}\frac{[\mathbf{p}\times\hat{\mathbf{q}}]^4}{(\mathbf{p}-\mathbf{q})^4+(p_1-q_1)^2} - \frac{1}{P_0(\mathbf{q})}\frac{[\mathbf{p}\times\hat{\mathbf{q}}]^4}{(\mathbf{p}-\mathbf{q})^4}\right](d\mathbf{q})$$

where dimensionless polarization operators are defined according to

$$\Pi^{\sigma=0}_{\mathbf{q}} = \frac{d_c T}{\varkappa^2}\int \frac{[\mathbf{k}\times\hat{\mathbf{q}}]^4 (d\mathbf{k})}{k^4(\mathbf{q}-\mathbf{k})^4} = \frac{d_c T}{\varkappa\sigma} P_0\left(\sqrt{\frac{\varkappa}{\sigma}}\mathbf{q}\right) = \frac{3}{16\pi}\frac{d_c T}{\varkappa^2}\frac{1}{q^2},$$
$$\Pi_{\mathbf{q}} = \frac{d_c}{T}\int \frac{T^2[\mathbf{k}\times\hat{\mathbf{q}}]^4 (d\mathbf{k})}{(\varkappa k^4+\sigma k_1^2)(\varkappa(\mathbf{q}-\mathbf{k})^4+\sigma(q_1-k_1)^2)} = \frac{d_c T}{\varkappa\sigma} P_1\left(\sqrt{\frac{\varkappa}{\sigma}}\mathbf{q}\right).$$

Dimensionless polarization operator for uniaxial stress $P_1(\mathbf{q})$ is calculated in the appendix.

Answer Since $\delta K_\beta^{[0]} \sim \frac{d_c^2}{8\pi}$, $d_c \to \infty$, the Poisson ratio in the first non–trivial order may be written as

$$\begin{aligned}\nu = -\frac{\delta K_2^{[0]} + \delta K_2^{[1]} + \ldots}{\delta K_1^{[0]} + \delta K_1^{[1]} + \ldots} &= -\frac{\delta K_2^{[0]}}{\delta K_1^{[0]}}\left[1 + \frac{\delta K_2^{[1]}}{\delta K_2^{[0]}} - \frac{\delta K_1^{[1]}}{\delta K_1^{[0]}} + \ldots\right] \\ &= -\frac{\delta K_2^{[0]}}{\delta K_1^{[0]}}\left[1 + \frac{8\pi}{d_c^2}\frac{\varkappa}{T}\left(\delta K_2^{[1]} - \delta K_1^{[1]}\right) + \ldots\right] \\ &= -\frac{1}{1+2\alpha}\left[1 + \frac{c}{d_c^2} + \mathcal{O}\left(\frac{1}{d_c^3}\right)\right]\end{aligned}$$

where the number $c = 8\pi(\#_2^{(0,0)} - \#_1^{(0,0)})$ is of interest.

$$\frac{c}{16\pi} = \int_{\mathbf{q},\mathbf{k}}\left[\frac{(k_2^2-k_1^2)[\mathbf{k}\times\hat{\mathbf{q}}]^4}{P_1(\mathbf{q})(k^4+k_1^2)^2((\mathbf{k}-\mathbf{q})^4+(k_1-q_1)^2)} - \frac{(k_2^2-k_1^2)[\mathbf{k}\times\hat{\mathbf{q}}]^4}{P_0(\mathbf{q})(k^4+k_1^2)^2(\mathbf{k}-\mathbf{q})^4}\right]$$

Such an integral may be computed numerically with $P_0(\mathbf{q}) = \frac{3}{16\pi}q^{-2}$ and the expression for $P_1(\mathbf{q})$ is provided in the appendix. The answer is $c = -0.56 \pm 0.01$, which allows me to conclude that the true expression for the absolute Poisson ratio in the non–linear regime $\sigma \ll \sigma_*$ is not given by a simple expression such as $1/(1+2\alpha)$, but a more complex function.

In the next section I relate the critical exponent α to its approximate (self–consistent) value α_{sc} in order to find the Poisson ratio up to $1/d_c^2$.

Chapter 4
Numerical simulations

4.1 Overview

In this section I sum up and give brief comments on recent published numerical studies of the universal regime of the model (2.2). Papers from the previous century are added selectively; for a more complete list the reader is directed to the recent review [5, Fig. 1]. Some notions in this

Year	Author	Method	p_8	Size, L^2	η	$\nu_{\sigma=0}$
1996	[17] Bowick	Hausdorff dim Space correlator	$\approx .4$ $\approx .4$	128^2	$.72 \pm .12$ $\approx .6$	
1997	[18] Falcioni *et al*	Space correlator	$\approx .4$	128^2	$\approx .62$	$-.3 \pm 0.06$
2001	[4] Bowick *et al*	Self–avoiding		95^2		$-.37 \pm 0.6$
2009	[19] Los *et al*	Molecular dynamics	–	190^2	$\approx .85$	
2012	[20] Troster	Fourier correlator Fourier Green	.44	$640^2(40^2)$	$.795 \pm .01$ $\approx .761$	
2016	[21] Los *et al*	Space correlator	.4	195^2	≈ 0.84	$+.275$
2019	[#] Saykin	Fourier Green	.3	360^2	$.78 \pm .02$	$-.76 \pm .05$

Table 4.1: Selected reports on numerical values of critical indices.

table require additional clarification. Most of the papers use Monte Carlo simulations of the action (2.2), some work in $\mathbf{r}$–space, some in Fourier $\mathbf{k}$–space, some papers study the behavior of the Green function itself, some study the correlator

$$K_{\alpha\beta} = \int d\mathbf{x}(\partial_\alpha \mathbf{r} \cdot \partial_\beta \mathbf{r})$$

instead. Column p_8 contains information about the interaction force. It is useful to measure it in the dimensionless number

$$p_8 = p_* a = a\sqrt{\frac{3}{16\pi}\frac{Y_0 T}{\varkappa^2}},$$

where I use $a = 2.46$ Å for the graphene lattice constant, $Y_0 \simeq 22$ eV·Å^{-2} and $\varkappa \simeq 1.1$ eV. For $T = 300$ K it gives $p_8 = .41$.

One can see that there are many discrepancies between the reported data, which calls for an explanation. Below I reproduce the method used in [20] and calculate η and the Poisson ratio $\nu_{\sigma=0}$.

4.2 Simulation description

I use the Metropolis–Hastings algorithm and run Monte–Carlo simulations with the weight function $e^{-F/T}$ and I use the (3.1) expression with $\sigma = 0$, which in dimensionless form looks as follows.

$$\mathcal{F} = \sum_{\mathbf{q}\neq 0} \left\{ \frac{1}{2} \left(\frac{2\pi q}{L} \right)^4 \left| h_{[\mathbf{q}]} \right|^2 + \frac{2\pi}{3} \frac{p_8^2}{L^2} \left| S_{[\mathbf{q}]} \right|^2 \right\}$$

with the following function

$$S_{[\mathbf{q}]} = \sum_{\mathbf{k}\neq 0} \left[\frac{2\pi}{L} \mathbf{k} \times \hat{\mathbf{q}} \right]^2 \left(h_{[\mathbf{k}]} \cdot h_{[\mathbf{k}+\mathbf{q}]} \right)$$

and all momenta replaced by sin–value

$$\frac{2\pi}{L} k_1 \mapsto \sin\left(\frac{2\pi}{L} k_1 \right), \qquad q^2 \mapsto 4 \sin^2\left(\frac{\pi}{L} q_1 \right) + 4 \sin^2\left(\frac{\pi}{L} q_2 \right).$$

Vector product is defined by its value in the first Brillouin zone and continued from there by periodicity.

Running $\sim 10^5$ Monte–Carlo steps I find the Green function

$$\mathcal{G}_{\mathbf{p}} = \left\langle \left| h_{[\mathbf{p}]} \right|^2 \right\rangle_{\text{MC}} = \frac{1}{\varkappa_p} \left(\frac{2\pi p}{L} \right)^{-4}.$$

I choose the phase space exploration step so that the acceptance rate varies between 35% and 55%. As is mentioned in [20] the step must be momentum–dependent and could be heuristically chosen from the relation $d_{[\mathbf{k}]} \sim h_{[\mathbf{k}]}$.

From the $\mathcal{G}_{\mathbf{k}}$ dependence I extract the value of the critical exponent η. I present the results in Figure 13 and Figure 14. As can be seen the smallest wave vectors behave inconsistently, which may be called a finite–size effect [20]. That's why I only use the third and higher momenta for the fit. I report the value

$$\eta = 0.78 \pm 0.02.$$

That value is in good agreement with the numerical paper [20] and the «second–order» self–consistent approximation [6].

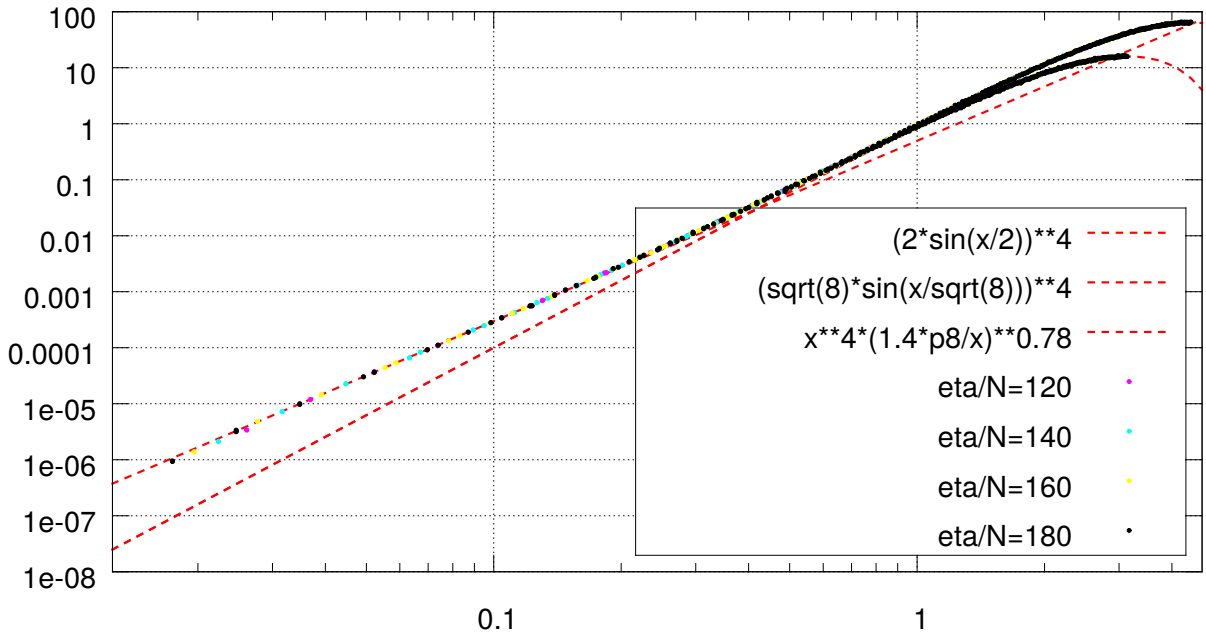


Figure 13: Log–log plot of $\mathcal{G}_{\mathbf{q}}^{-1}$ for different lattice sizes $L = 200$, 240, 280, 320, 360.

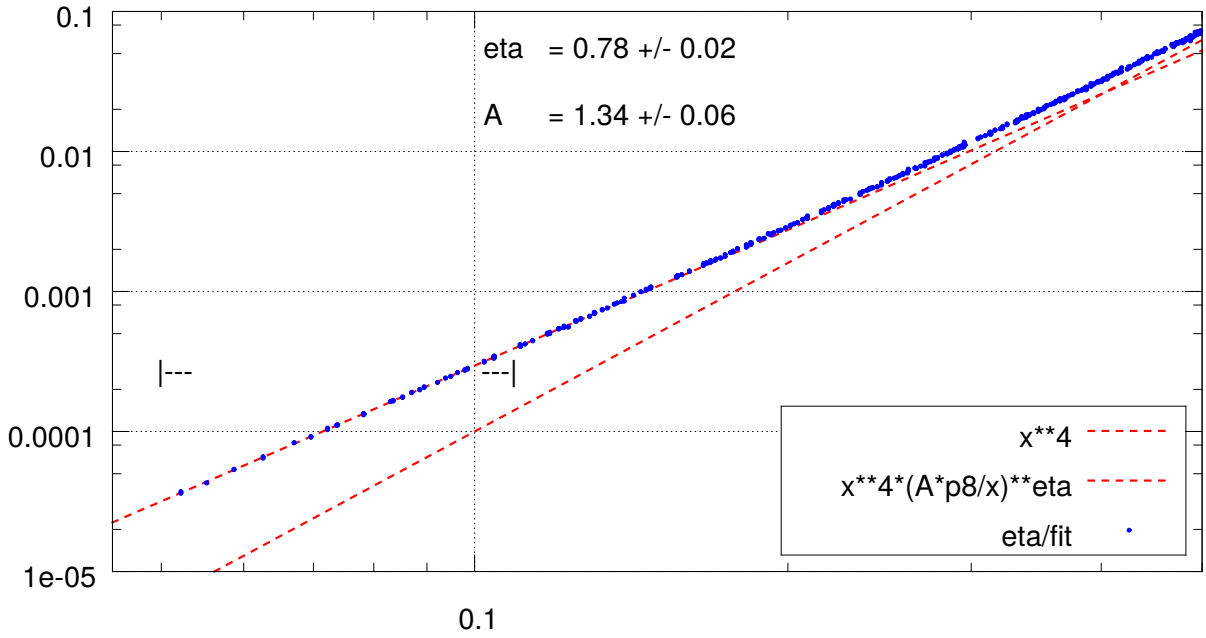


Figure 14: Close–up view. Fit region is marked. Presented lattice sizes $L = 200$, 220, 240, 260, 280, 300, 320, 340, 360.

Poisson ratio I also calculate the value of the Poisson ratio in the non–linear regime as

$$\nu_{\sigma=0} = -\frac{\sum_{|\mathbf{q}|<q_8} \left\langle K^x_{[\mathbf{q}]} K^y_{[\mathbf{q}]} \right\rangle - \left\langle K^x_{[\mathbf{q}]} \right\rangle \left\langle K^y_{[\mathbf{q}]} \right\rangle}{\sum_{|\mathbf{q}|<q_8} \left\langle K^x_{[\mathbf{q}]} K^x_{[\mathbf{q}]} \right\rangle - \left\langle K^x_{[\mathbf{q}]} \right\rangle^2}, \qquad K^\alpha_{[\mathbf{q}]} = \sin^2\left(\frac{2\pi}{L} k_\alpha\right) \left|h_{[\mathbf{q}]}\right|^2$$

and report the value

$$\nu_{\sigma=0} = -0.76 \pm 0.05.$$

4.2.1 Algorithm

The idea of the effective calculation is expounded in [22]. The key idea is to keep the $S_{[\mathbf{q}]}$ and $dS_{[\mathbf{q}]}$ arrays and evaluate only $dS_{[\mathbf{q}]}$ each Monte–Carlo step. Here is the listing of the code that does such evaluation.

```
for (q1 = 0; q1 < L; q1++)
  for (q2 = 0; q2 < L; q2++) {
    if (!q1 && !q2) continue;
    double p = sin[k1]*sin[q2]-sin[k2]*sin[q1]; p *= p;
    double kq = sin[k2+q2]*sin[q1]-sin[k1+q1]*sin[q2]; kq *= kq;
    double qk = sin[k1-q1]*sin[q2]-sin[k2-q2]*sin[q1]; qk *= qk;
    complex s = p*conj(h[k1+q1][k2+q2])*z;
    s += kq*h[-k1-q1][-k2-q2]*z;
    s += qk*h[k1-q1][k2-q2]*conj(z);
    s +=  p*conj(h[q1-k1][q2-k2])*conj(z);
    if (!((q1+2*k1)%L) && !((q2+2*k2)%L))
      s += p*z*z*d;
    if (!((q1-2*k1+2*L)%L) && !((q2-2*k2+2*L)%L))
      s += p*conj(z)*conj(z)*d;
    s *= d/Q[q1][q2];
    dS[q1][q2] = s;
    w += creal((2*S[q1][q2]+s)*conj(s));
}
```

Main body of the simulate step function is printed on the next page. Here I note these listings are not one–to–one exact to a working code. The reader may find the source code at github.com/saykind.

```
#include <complex.h>
#include <omp.h>

int simulate(complex **h, complex **S, complex **dS, double **g) {
  int L = 2*N+1;
  int k1 = rand()%L-N, k2 = rand()%L-N, q1, q2;
  if (!k1 && !k2) {simulate(h, S, dS, g); return 0;}
  double w = 0, A = Q[k1][k2], d = 2.6/A/pow(1+Y/A,.13); A *= A;

  complex z = (1.*rand()/RAND_MAX-.5) + (1.*rand()/RAND_MAX-.5)*I;
  #pragma omp parallel for collapse(2) reduction(+:w)
  for (q1 = 0; q1 < L; q1++)
    for (q2 = 0; q2 < L; q2++) {
      if (!q1 && !q2) continue;
      /* Calculation of s=dS[q1][q2] */
      dS[q1][q2] = s;
      w += creal((2*S[q1][q2]+s)*conj(s));
  }
  w *= -Y/L/L;
  w -= A*creal((2*h[k1][k2] + d*z)*conj(z))*d;
  if (w > log(1.*rand()/RAND_MAX)) {
    h[k1][k2] += d*z;
    h[(L-k1)%L][(L-k2)%L] += d*conj(z);
    #pragma omp parallel for collapse(2)
    for (q1 = 0; q1 < L; q1++)
      for (q2 = 0; q2 < L; q2++)
        S[q1][q2] += dS[q1][q2];
  }
  if (c && g) {
    double a = creal(h[k1][k2]*conj(h[k1][k2]));
    g[k1][k2] += a;
    g[(L-k1)%L][(L-k2)%L] += a;
  }
  return 0;
}
```

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Chapter 5

Summary

5.1 Results

5.1.1 Critical exponents

I have found the critical exponent η and its derivative $\alpha = \frac{\eta}{2-\eta}$ that characterize the universal regime (the system forgets about the material–dependent characteristics μ, λ, $\varkappa$) of a large enough two–dimensional crystal under not strong stress ($\sigma_L \ll \sigma \ll \sigma_*$) perturbatively in $1/d_c$ up to the first non–trivial[1] order.

$$\begin{aligned}
\eta &= \frac{2}{d_c} + \frac{73 - 68\zeta(3)}{27}\frac{1}{d_c^2} + \mathcal{O}\big(d_c^{-3}\big)\,, \\
&= \frac{2}{d_c} - \frac{.324}{d_c^2} + \mathcal{O}\big(d_c^{-3}\big)\,. \\
\alpha &= \frac{1}{d_c} + \frac{127 - 68\zeta(3)}{54}\frac{1}{d_c^2} + \mathcal{O}\big(d_c^{-3}\big)\,, \\
&= \frac{1}{d_c} + \frac{.838}{d_c^2} + \mathcal{O}\big(d_c^{-3}\big)\,.
\end{aligned}$$

In particular, my calculations hint that the true value of η and α may be considerably different from the values given by the self–consistent screening approximation $\eta_{\text{SCSA}} = \frac{4}{1+\sqrt{15}} \approx 0.821$ and $\alpha_{\text{SCSA}} = \frac{1}{7}(1+\sqrt{15}) \approx 0.696$.

$$\begin{aligned}
\eta &= \eta_{\text{SCSA}} - \frac{68\zeta(3) - 46}{108}\eta_{\text{SCSA}}^2 + \mathcal{O}\big(\eta_{\text{SCSA}}^3\big)\,, \\
&= \eta_{\text{SCSA}} - .331\eta_{\text{SCSA}}^2 + \mathcal{O}\big(\eta_{\text{SCSA}}^3\big)\,. \\
\alpha &= \alpha_{\text{SCSA}} - \frac{34\zeta(3) - 23}{27}\alpha_{\text{SCSA}}^2 + \mathcal{O}\big(\alpha_{\text{SCSA}}^3\big)\,, \\
&= \alpha_{\text{SCSA}} - .662\alpha_{\text{SCSA}}^2 + \mathcal{O}\big(\alpha_{\text{SCSA}}^3\big)\,.
\end{aligned}$$

In order to make even a very bold estimate of the actual values of η and α from these expressions one has to know whether one deals with a sign–alternating series or not. I make a wild guess that the next term goes with a positive sign, and estimate $\eta \approx .7$ and $\alpha \approx .5$.

[1] Before, only the leading term $\eta \sim 2/d_c$ and the SCSA approximation were known.

5.1.2 Absolute Poisson ratio

In the aforementioned universal regime, the Poisson ratio also becomes a universal number.

$$\begin{aligned} -\nu &= 1 - \frac{2}{d_c} + \frac{1.76}{d_c^2} + \mathcal{O}\big(d_c^{-3}\big)\,, \\ &= 1 - \eta + .36\eta^2 + \mathcal{O}\big(\eta^3\big)\,, \\ &= 1 - \eta_{\text{SCSA}} + .69\eta_{\text{SCSA}}^2 + \mathcal{O}\big(\eta_{\text{SCSA}}^3\big)\,, \\ &= 1 - 2\alpha + 3.44\alpha^2 + \mathcal{O}\big(\alpha^3\big)\,, \\ &= 1 - 2\alpha_{\text{SCSA}} + 4.12\alpha_{\text{SCSA}}^2 + \mathcal{O}\big(\alpha_{\text{SCSA}}^3\big)\,. \end{aligned}$$

I would also like to make a guess about the actual value of ν, so I take my chances and suppose that the series is alternating in sign, then I would say $\nu \approx -.4$ might be a good guess.

5.1.3 Numerical results

Here I report the result of Monte–Carlo simulations.

$$\eta = 0.78 \pm 0.02, \qquad \nu_{\sigma=0} = -0.76 \pm 0.05.$$

Please note that the value of the Poisson ratio at zero stress does not have much in common with the value ν, $\sigma_L \ll \sigma \ll \sigma_*$ discussed in the main paper, as was pointed out in [10].

5.2 Conclusion

Here I give answers to the questions stated at the beginning of the paper.

1. Critical exponent α from anomalous Hooke's law could differ drastically (10%) from the SCSA value [9]. There is no theoretical reason to believe it is close to $\alpha_{\text{SCSA}} \approx .7$. Numerical simulations suggest that the value is somewhere in the range $\alpha = .64 \pm .2$.

2. The value of the Poisson ratio ν in the universal regime is an independent critical index. Similarly, there is no reason to believe that either the differential or absolute Poisson ratio values are close to their SCSA approximations.

Appendix A

Calculation details

A.1 Second order non–SCSA contributions

Here I give the details of the calculation of the second–order contributions to the self energy that are present in (3.10, 3.11, 3.12, 3.13).

Second–order. Crossed diagram Here goes the diagram depicted in Figure 12–12*c*.

$$T\Sigma_{\mathbf{p}}^{(2c)} = \int \frac{Y_0^2 T^2 (d\mathbf{q}d\mathbf{k})[\mathbf{p}-\mathbf{q}\times\mathbf{k}]^2[\mathbf{p}-\mathbf{k}\times\mathbf{q}]^2[\mathbf{p}\times\mathbf{k}]^2[\mathbf{p}\times\mathbf{q}]^2}{\varkappa^3(q^4+\frac{p_*^2}{2}q^2)(k^4+\frac{p_*^2}{2}k^2)(\mathbf{p}-\mathbf{q})^4(\mathbf{p}-\mathbf{q}-\mathbf{k})^4(\mathbf{p}-\mathbf{k})^4} = \frac{\varkappa p_*^4}{d_c^2}\mathcal{S}^{(2c)}\left(\frac{\sqrt{2}p}{p_*}\right),$$

$$\begin{aligned}\mathcal{S}^{(2c)}(p) &= \left(\frac{16\pi}{3}\right)^2 \int \frac{(d\mathbf{q})[\mathbf{p}\times\mathbf{q}]^2}{(q^4+q^2)(\mathbf{p}-\mathbf{q})^4}\int \frac{[\mathbf{p}\times\mathbf{k}]^2[\mathbf{p}-\mathbf{q}\times\mathbf{k}]^2[\mathbf{p}-\mathbf{k}\times\mathbf{q}]^2}{(k^4+k^2)(\mathbf{p}-\mathbf{q}-\mathbf{k})^4(\mathbf{p}-\mathbf{k})^4}(d\mathbf{k}) \\ &\sim \left(\frac{16\pi}{3}\right)^2 p^4 \int \frac{(d\mathbf{q})q^2[\hat{\mathbf{p}}\times\hat{\mathbf{q}}]^2}{(1+q^{-2})(\mathbf{p}-\mathbf{q})^4}\int \frac{[\hat{\mathbf{p}}\times\hat{\mathbf{k}}]^2[\hat{\mathbf{k}}\times\hat{\mathbf{q}}]^4}{(\mathbf{k}-\mathbf{q})^4}(d\mathbf{k}), \qquad p\ll 1.\end{aligned}$$

I use identity (3.7) and two more expressions

$$\left\langle \frac{[\hat{\mathbf{p}}\times\hat{\mathbf{k}}]^2[\hat{\mathbf{k}}\times\hat{\mathbf{q}}]^4}{(\mathbf{k}-\mathbf{q})^4}\right\rangle_{\hat{\mathbf{k}}} = \frac{3/16}{\max\{k^4,q^4\}} + \left[\frac{4}{\max\{k^4,q^4\}} - \frac{5\min\{k^2,q^2\}}{\max\{k^6,q^6\}}\right]\frac{\cos[2\widehat{\mathbf{pq}}]}{32},$$

$$\left\langle \frac{[\hat{\mathbf{p}}\times\hat{\mathbf{q}}]^2}{(\mathbf{p}-\mathbf{q})^4}\cos[2\widehat{\mathbf{pq}}]\right\rangle_{\hat{\mathbf{k}}} = \frac{3\min\{p^2,q^2\}-\max\{p^2,q^2\}}{4\max\{p^4,q^4\}|p^2-q^2|}.$$

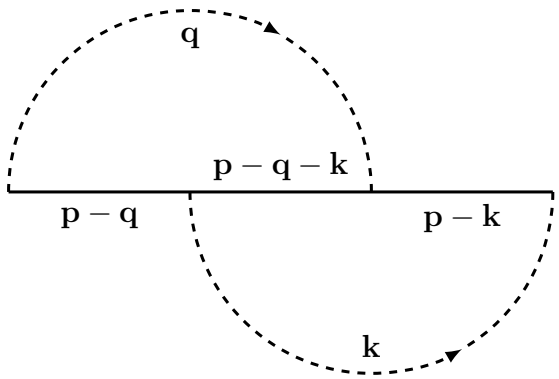

Figure 15: Crossed diagram

Changing the variables $q^2 \mapsto q$, $k^2 \mapsto k$, I come to

$$\begin{aligned}\mathcal{S}^{(2c)}(p) &= \frac{1}{12}\frac{p^4}{4}\int_0^\infty \frac{qdqdk}{1+q^{-1}}\left(\frac{6}{\max\{k^2,q^2\}}\cdot\frac{4}{3}\frac{1}{\max\{p^2,q\}|p^2-q|}+\right.\\ &\left.+\left[\frac{4}{\max\{k^2,q^2\}}-5\frac{\min\{k,q\}}{\max\{k^3,q^3\}}\right]\cdot\frac{2}{3}\frac{3\min\{p^2,q\}-\max\{p^2,q\}}{\max\{p^4,q^2\}|p^2-q|}\right)\\ &= \frac{p^4}{4}\int_0^\infty \frac{dq}{1+q^{-1}}\left(\frac{4/3}{\max\{p^2,q\}|p^2-q|}+\frac{3\min\{p^2,q\}-\max\{p^2,q\}}{6\max\{p^4,q^2\}|p^2-q|}\right)\end{aligned}$$

This integral is not hard to calculate exactly. Finally,

$$\mathcal{S}^{(2c)}(p) \sim -\frac{7}{3}\left(\frac{p}{\sqrt{2}}\right)^4 \ln p + \text{const.}$$

That is the result (3.10).

Second–order. Pill diagram Here is another one.

$$\begin{aligned}T\Sigma_{\mathbf{p}}^{(2d)} &= -\frac{d_c}{2}\frac{Y_0^3T^3}{\varkappa^5}\int\frac{(d\mathbf{q}d\mathbf{r}d\mathbf{k})[\mathbf{p}\times\mathbf{q}]^4[\mathbf{k}\times\mathbf{q}]^2[\mathbf{k}\times\mathbf{r}]^2[\mathbf{k}-\mathbf{r}\times\mathbf{q}]^2[\mathbf{k}-\mathbf{q}\times\mathbf{r}]^2}{(q^4+\frac{p_*^2}{2}q^2)^2(r^4+\frac{p_*^2}{2}r^2)(\mathbf{p}-\mathbf{q})^4k^4(\mathbf{k}-\mathbf{q})^4(\mathbf{k}-\mathbf{r})^4(\mathbf{k}-\mathbf{q}-\mathbf{r})^4}\\ &= \frac{\varkappa p_*^4}{d_c^2}\mathcal{S}^{(2d)}\left(\sqrt{2}\frac{p}{p_*}\right),\\ \mathcal{S}^{(2d)}(p) &= -\left(\frac{16\pi}{3}\right)^3\int\frac{[\mathbf{p}\times\mathbf{q}]^4q^2(d\mathbf{q})}{(q^4+q^2)^2(\mathbf{p}-\mathbf{q})^4}\int\frac{(d\mathbf{r}d\mathbf{k})[\mathbf{k}\times\hat{\mathbf{q}}]^2[\mathbf{k}\times\mathbf{r}]^2[\mathbf{k}-\mathbf{r}\times\hat{\mathbf{q}}]^2[\mathbf{k}-\hat{\mathbf{q}}\times\mathbf{r}]^2}{(\cancel{q^2r^4}+r^2)k^4(\mathbf{k}-\hat{\mathbf{q}})^4(\mathbf{k}-\mathbf{r})^4(\mathbf{k}-\hat{\mathbf{q}}-\mathbf{r})^4}\end{aligned}$$

$$\mathcal{S}^{(2d)}(p) \sim -\left(\frac{16\pi}{3}\right)^2\frac{f}{2}\int_0^\infty\frac{dqq}{(q^2+q)^2}\min\{p^4,q^2\} \sim \left(\frac{16\pi}{3}\right)^2 fp^4\ln p.$$

So I am interested in calculating number f.

$$f = \int\frac{[\hat{\mathbf{k}}\times\hat{\mathbf{q}}]^2[\hat{\mathbf{k}}\times\hat{\mathbf{r}}]^2[\hat{\mathbf{r}}\times\hat{\mathbf{q}}]^2}{(\mathbf{k}-\hat{\mathbf{q}})^4(\mathbf{k}-\mathbf{r})^2(\mathbf{r}-\hat{\mathbf{q}})^4}[\mathbf{k}-\hat{\mathbf{q}}\times\mathbf{r}-\hat{\mathbf{q}}]^2(d\mathbf{r}d\mathbf{k})$$

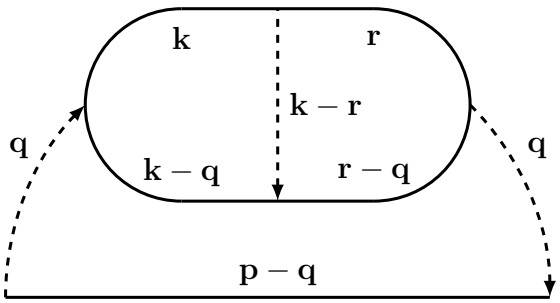

Figure 16: The Eye of Sauron diagram

In order to average over angles I change the variables to $z = \exp\left[i\widehat{\mathbf{kq}}\right]$, $w = \exp\left[i\widehat{\mathbf{rq}}\right]$. I find several potential residues $z = 0$, k, $1/k$, kw/r, rw/k. Depending on parameters k, r they either lie inside unit circle $|z| = 1$ or not. After evaluating integrals over z I come to expressions with potential residue at $w = 0$, r, $1/r$, k^2/r, r/k^2. All in all, I have six different cases depending on relation between $k > 1$, $r > 1$, $k > r$. It so happens that relation $k^2 > r$ does not change the answer. Altogether contributions give $f = (1/2)(3/16\pi)^2$. Thus,

$$d_c^2 \frac{T\Sigma_{\mathbf{p}}^{(2d)}}{\varkappa p^4} \sim 2\ln\left[\frac{p}{p_*}\right] + \text{const.}$$

That is the result (3.11). Results (3.12), (3.13) are obtained in a similar manner.

A.2 Polarization operator

Here I calculate the polarization operator for the case of uniaxial stress $\sigma_1 = \sigma$, $\sigma_2 = 0$.

$$\Pi_{\mathbf{q}} = \frac{d_c T}{\varkappa\sigma}\int \frac{[\mathbf{k}\times\hat{\mathbf{q}}]^2}{\mathbf{k}^4+k_1^2}\frac{[(\mathbf{q}-\mathbf{k})\times\hat{\mathbf{q}}]^2}{(\mathbf{q}-\mathbf{k})^4+(q_1-k_1)^2}(d\mathbf{k}) = \frac{d_c T}{\varkappa\sigma}P\left(\sqrt{\frac{\varkappa}{\sigma}}\mathbf{q}\right),$$
$$P(\mathbf{q}) = \int (d\mathbf{k}) f_{\mathbf{k}} f_{\mathbf{q}-\mathbf{k}} = \int d^2\mathbf{x}\, e^{-i\mathbf{q}\mathbf{x}} f^2(\mathbf{x}), \qquad f(\mathbf{x}) = \int \frac{[\mathbf{k}\times\hat{\mathbf{q}}]^2}{k^4+k_1^2} e^{i\mathbf{k}\mathbf{x}}(d\mathbf{k}).$$

Function $f(\mathbf{x})$ may be calculated in coordinates. Let me define

$$f^{(a,b)}(\mathbf{x}) = \int \frac{k_1^a k_2^b}{k^4+k_1^2} e^{i\mathbf{k}\mathbf{x}}(d\mathbf{k}), \qquad a,b\in\mathbb{N}_0,\ \ a+b\neq 0,$$
$$f^{(0,0)}(\mathbf{x}) = \int \frac{e^{i\mathbf{k}\mathbf{x}}-1}{k^4+k_1^2}(d\mathbf{k}),$$

then the function I'm interested in is given by $f(\mathbf{x}) = -[\hat{\mathbf{q}}\times\nabla]^2 f^{(0,0)}(\mathbf{x})$. Unfortunately, I don't know how to calculate $f^{(0,0)}$. Fortunately, $f^{(1,0)}$ and $f^{(2,0)}+f^{(0,2)}$ are possible to calculate.

$$f^{(1,0)}(\mathbf{x}) = \sum_{\pm}\frac{\mp}{2i}\int \frac{e^{i\mathbf{k}\mathbf{x}}(d\mathbf{k})}{k^2\pm ik_1} = \sum_{\pm}\frac{\mp}{2i}\int\frac{kdk}{2\pi}\frac{J_0(kx)}{k^2+\frac{1}{4}}e^{\pm x_1/2} = \frac{i}{2\pi}\operatorname{sh}\frac{x_1}{2}K_0\frac{x}{2}.$$
$$f^{(2,0)}(\mathbf{x})+f^{(0,2)}(\mathbf{x}) = \sum_{\pm}\frac{1}{2}\int \frac{e^{i\mathbf{k}\mathbf{x}}(d\mathbf{k})}{k^2\pm ik_1} = \sum_{\pm}\frac{1}{2}\int\frac{kdk}{2\pi}\frac{J_0(kx)}{k^2+\frac{1}{4}}e^{\pm x_1/2} = \frac{1}{2\pi}\operatorname{ch}\frac{x_1}{2}K_0\frac{x}{2}.$$

From these expressions, with the help of the property $f^{(a+n,b+m)} = (-i\partial_1)^n(-i\partial_2)^m f^{(a,b)}$ I am able to find the following functions.

$$f^{(2,0)} = \frac{1}{4\pi}\left[\operatorname{ch}\frac{x_1}{2}K_0\frac{x}{2} - \frac{x_1}{x}\operatorname{sh}\frac{x_1}{2}K_1\frac{x}{2}\right],$$
$$f^{(1,1)} = -\frac{1}{4\pi}\frac{x_2}{x}\operatorname{sh}\frac{x_1}{2}K_1\frac{x}{2},$$
$$f^{(0,2)} = \frac{1}{4\pi}\left[\operatorname{ch}\frac{x_1}{2}K_0\frac{x}{2} + \frac{x_1}{x}\operatorname{sh}\frac{x_1}{2}K_1\frac{x}{2}\right].$$

Now I need to multiply different functions in $\mathbf{x}$–space and perform the inverse Fourier transform. Let me define $F^{(a,b;c,d)}(\mathbf{x}) = f^{(a,b)}(\mathbf{x})f^{(c,d)}(\mathbf{x})$ and $F^{(a,b)}(\mathbf{x}) = f^{(a,b)}(\mathbf{x})f^{(a,b)}(\mathbf{x})$, then the polarization operator is given by

$$P(\mathbf{q}) = \hat{q}_1^4 F_{\mathbf{q}}^{(0,2)} + 4\hat{q}_1^2\hat{q}_2^2 F_{\mathbf{q}}^{(1,1)} + \hat{q}_2^4 F_{\mathbf{q}}^{(0,2)} + 2\hat{q}_1^2\hat{q}_2^2 F_{\mathbf{q}}^{(0,2,2,0)} - 4\hat{q}_1^3\hat{q}_2 F_{\mathbf{q}}^{(0,2,1,1)} - 4\hat{q}_1\hat{q}_2^3 F_{\mathbf{q}}^{(2,0,1,1)}. \quad \text{(A.1)}$$

To evaluate these integrals I use the following identities.

$$\int_0^\infty xK_0^2\left(\frac{x}{2}\right)J_0(qx)dx = \frac{2\operatorname{arcsh} q}{q\sqrt{1+q^2}} \equiv 2A(q),$$
$$\int_0^\infty xK_0\left(\frac{x}{2}\right)K_1\left(\frac{x}{2}\right)J_1(qx)dx = \frac{2\operatorname{arcsh} q}{\sqrt{1+q^2}} \equiv 2qA(q),$$
$$\int_0^\infty K_1^2\left(\frac{x}{2}\right)\left[J_1(qx)-\frac{qx}{2}\right] = 2\left[q-\sqrt{1+q^2}\operatorname{arcsh} q\right] \equiv -2q(1+q^2)A(q)+2q.$$

Let me show how to calculate these integrals one by one. Let me begin with

$$\begin{aligned}
F_{\mathbf{q}}^{(1,1)} &= \int \frac{d^2\mathbf{x}}{(4\pi)^2}e^{-i\mathbf{qx}}\left(\frac{x_2}{x}\operatorname{sh}\frac{x_1}{2}K_1\frac{x}{2}\right)^2 \\
&= \sum_{\pm}\int \frac{d^2\mathbf{x}}{(8\pi)^2}e^{-i\mathbf{qx}}\left(\frac{x_2}{x}\right)^2(e^{\pm x_1}-1)K_1^2\frac{x}{2} \\
&= \operatorname{Re}\int_0^\infty \frac{xdx}{16\pi}\frac{\partial_{q_2}}{x}\left[\hat{p}_2J_1(px)-\hat{q}_2J_1(qx)\right]K_1^2\frac{x}{2} \\
&= \frac{1}{8\pi}\operatorname{Re}\partial_2\left[q_2(1+q^2)A(q)-p_2(1+p^2)A(p)\right] \\
&= \frac{1}{8\pi}\left(\left[\hat{q}_1^2(1+q^2)+q_2^2\right]A(q)+\hat{q}_2^2-\operatorname{Re}\left\{\left[\hat{p}_1^2(1+p^2)+p_2^2\right]A(p)+\hat{p}_2^2\right\}\right).
\end{aligned}$$

where I have introduced vector $\mathbf{p} = (q_1+i, q_2)$. The following two are similar.

$$\begin{aligned}
F_{\mathbf{q}}^{(0,2)} &= \int \frac{d^2\mathbf{x}}{(4\pi)^2}e^{-i\mathbf{qx}}\left(\operatorname{ch}\frac{x_1}{2}K_0\frac{x}{2}+\frac{x_1}{x}\operatorname{sh}\frac{x_1}{2}K_1\frac{x}{2}\right)^2 \\
&= \sum_{\pm}\int \frac{d^2\mathbf{x}}{(8\pi)^2}e^{-i\mathbf{qx}}\left((e^{\pm x_1}+1)K_0^2\frac{x}{2}\pm 2\frac{x_1}{x}e^{\pm x_1}K_0\frac{x}{2}K_1\frac{x}{2}+\left(\frac{x_1}{x}\right)^2(e^{\pm x_1}-1)K_1^2\frac{x}{2}\right) \\
&= \operatorname{Re}\int_0^\infty \frac{xdx}{16\pi}\left(\left[J_0(px)+J_0(qx)\right]K_0^2\frac{x}{2}+2i\frac{\partial_{q_1}}{x}J_0(px)K_0\frac{x}{2}K_1\frac{x}{2}+\frac{\partial_{q_1}}{x}\left[\hat{p}_1J_1(px)-\hat{q}_1J_1(qx)\right]K_1^2\frac{x}{2}\right) \\
&= \frac{1}{8\pi}\operatorname{Re}\left\{A(p)+A(q)-2ip_1A(p)+\partial_1\left[q_1(1+q^2)A(q)-p_1(1+p^2)A(p)\right]\right\} \\
&= \frac{1}{8\pi}\left(\left[\hat{q}_2^2(1+q^2)+1+q_1^2\right]A(q)+\hat{q}_1^2-\operatorname{Re}\left\{\left[\hat{p}_2^2(1+p^2)-1+2ip_1+p_1^2\right]A(p)+\hat{p}_1^2\right\}\right).
\end{aligned}$$

The same way but with another sign is done the next one.

$$\begin{aligned}
F_{\mathbf{q}}^{(2,0)} &= \int \frac{d^2\mathbf{x}}{(4\pi)^2}e^{-i\mathbf{qx}}\left(\operatorname{ch}\frac{x_1}{2}K_0\frac{x}{2}-\frac{x_1}{x}\operatorname{sh}\frac{x_1}{2}K_1\frac{x}{2}\right)^2 \\
&= \frac{1}{8\pi}\operatorname{Re}\left\{A(p)+A(q)+2ip_1A(p)+\partial_1\left[q_1(1+q^2)A(q)-p_1(1+p^2)A(p)\right]\right\} \\
&= \frac{1}{8\pi}\left(\left[\hat{q}_2^2(1+q^2)+1+q_1^2\right]A(q)+\hat{q}_1^2-\operatorname{Re}\left\{\left[\hat{p}_2^2(1+p^2)-1-2ip_1+p_1^2\right]A(p)+\hat{p}_1^2\right\}\right).
\end{aligned}$$

The next one is also very similar to $F_{\mathbf{q}}^{(0,2)}$. Functions $F_{\mathbf{q}}^{(0,2;1,1)}$ and $F_{\mathbf{q}}^{(2,0;1,1)}$ are done similarly.

$$\begin{aligned}
F_{\mathbf{q}}^{(0,2;2,0)} &= \int \frac{d^2\mathbf{x}}{(4\pi)^2} e^{-i\mathbf{q}\mathbf{x}} \operatorname{ch}^2 \frac{x_1}{2} K_0^2 \frac{x}{2} - \left(\frac{x_1}{x}\right)^2 \operatorname{sh}^2 \frac{x_1}{2} K_1^2 \frac{x}{2} \\
&= \frac{1}{8\pi} \operatorname{Re} \left\{ A(p) + A(q) - \partial_1 \left[q_1(1+q^2)A(q) - p_1(1+p^2)A(p) \right] \right\} \\
&= \frac{1}{8\pi} \left(\left[-\hat{q}_2^2(1+q^2) + 1 - q_1^2 \right] A(q) - \hat{q}_1^2 - \operatorname{Re} \left\{ \left[-\hat{p}_2^2(1+p^2) - 1 - p_1^2 \right] A(p) - \hat{p}_1^2 \right\} \right).
\end{aligned}$$

Altogether, required functions are

$$\begin{aligned}
8\pi F_{\mathbf{q}}^{(0,2)} &= \left[\hat{q}_2^2 + q^2 + 1 \right] A(q) + \hat{q}_1^2 && - \operatorname{Re} \left\{ \left[\hat{p}_2^2 + p^2 - 1 + 2ip_1 \right] A(p) + \hat{p}_1^2 \right\}, \\
8\pi F_{\mathbf{q}}^{(2,0)} &= \left[\hat{q}_2^2 + q^2 + 1 \right] A(q) + \hat{q}_1^2 && - \operatorname{Re} \left\{ \left[\hat{p}_2^2 + p^2 - 1 - 2ip_1 \right] A(p) + \hat{p}_1^2 \right\}, \\
8\pi F_{\mathbf{q}}^{(1,1)} &= \left[\hat{q}_1^2 + q^2 \right] A(q) + \hat{q}_2^2 && - \operatorname{Re} \left\{ \left[\hat{p}_1^2 + p^2 \right] A(p) + \hat{p}_2^2 \right\}, \\
-8\pi F_{\mathbf{q}}^{(0,2;2,0)} &= \left[\hat{q}_2^2 + q^2 - 1 \right] A(q) + \hat{q}_1^2 && - \operatorname{Re} \left\{ \left[\hat{p}_2^2 + p^2 + 1 \right] A(p) + \hat{p}_1^2 \right\}, \\
-8\pi F_{\mathbf{q}}^{(0,2;1,1)} &= \hat{q}_1 \hat{q}_2 (1 - A(q)) && - \operatorname{Re} \left\{ \left[-\hat{p}_1 \hat{p}_2 + ip_2 \right] A(p) + \hat{p}_1 \hat{p}_2 \right\}, \\
-8\pi F_{\mathbf{q}}^{(2,0;1,1)} &= \hat{q}_1 \hat{q}_2 (A(q) - 1) && - \operatorname{Re} \left\{ \left[\hat{p}_1 \hat{p}_2 + ip_2 \right] A(p) - \hat{p}_1 \hat{p}_2 \right\},
\end{aligned}$$

with $A(q) = \operatorname{arcsh}(q)/q\sqrt{1+q^2}$ and $\mathbf{p} = (q_1 + i, q_2)$. That together with (A.1) sums up the answer for the polarization operator.

Asymptotics of Polarization operator I rewrite an expression separating different orders in q.

$$\begin{aligned}8\pi P(\mathbf{q}) = (1-4\hat{q}_1^2)\hat{q}_2^2\left[(A(q)-1) - \operatorname{Re}\left\{q^2p^{-2}(A(p)-1)\right\}\right] - \\ - 4\hat{q}_1^2\hat{q}_2^2(\hat{q}_2^2-\hat{q}_1^2)\cdot\hat{q}_1^{-1}\operatorname{Re}\left\{iqp^{-2}(A(p)-1)\right\} + \\ + \left[q^2A(q) - \operatorname{Re}p^2A(p)\right] + \\ + \left[A(q)+\operatorname{Re}A(p)\right] + \\ + 4\hat{q}_1^2\hat{q}_2^2\left[A(q)-\operatorname{Re}A(p)\right] - \\ - 2\left(\hat{q}_1^2-\hat{q}_2^2\right)\operatorname{Re}ip_1A(p) - \\ - 4\hat{q}_1^2\hat{q}_2^2\cdot(\hat{q}_1\hat{q}_2)^{-1}\operatorname{Re}ip_2A(p).\end{aligned}$$

with $A(q) = \operatorname{arcsh}(q)/q\sqrt{1+q^2}$ and $\mathbf{p} = (q_1+i, q_2)$.
With the help of `Mathematica` software, I find

$$P(\mathbf{q}) \sim \frac{3}{16\pi q^2} + \frac{3}{64\pi q^4} - (3-2\cos 2\varphi)\frac{\ln 2q}{16\pi q^4} + \frac{\cos 4\varphi - 32\cos 2\varphi}{192\pi q^4} + o\left(\frac{1}{q^4}\right), \quad q\to\infty.$$

From here it follows that

$$L_1(\mathbf{q}) = \frac{1}{4}\left(2+q\frac{\partial}{\partial q}\right)P(\mathbf{q}) \sim -\frac{1}{4}\frac{9}{32\pi q^4} + (3-2\cos 2\varphi)\frac{\ln 2q}{32\pi q^4} + \frac{44\cos 2\varphi - \cos 4\varphi}{4\times 96\pi q^4}, \quad q\to\infty$$

$$\langle L_1(\mathbf{q})\rangle_{\hat{\mathbf{q}}} \sim \frac{-9/4+3\ln 2q}{32\pi q^4}.$$

Comment Earlier I've used the following expression for asymptotics of L_α.

$$\begin{aligned}\langle L_1(\mathbf{q})\rangle_{\hat{\mathbf{q}}} &\sim \frac{1}{q^4}\int \frac{k_1^2(d\mathbf{k})}{(k^4+k_1^2/q^2)^2}\left\langle\frac{[\mathbf{k}\times\hat{\mathbf{q}}]^4}{(\mathbf{k}-\hat{\mathbf{q}})^4}\right\rangle_{\hat{\mathbf{q}}} = \int_0^\infty \frac{dkf(k)}{2\pi k}\left\langle\frac{\hat{k}_1^2}{(k^2+\hat{k}_1^2/q^2)^2}\right\rangle_{\hat{\mathbf{k}}} \\ &= \frac{1}{q^4}\int_0^\infty \frac{dk}{4\pi q^2}\frac{f(k)}{k^2(1/q^2+k^2)^{3/2}} \sim \frac{-9/4+3\ln 2q}{32\pi q^4}, \quad q\to\infty.\end{aligned}$$

Which somehow justifies the validity of the used way to calculate asymptotics in the main text.

Small argument At $q \ll 1$ the asymptotic of $P(\mathbf{q})$ is highly anisotropic.

$$P(\mathbf{q}) \sim \frac{1}{8}\begin{cases}\frac{1}{\pi}, & q_2^2 \gg q_1, \\ \hat{q}_1^4|q_1|^{-1/2}, & q_2^2 \ll q_1,\end{cases} \qquad \sqrt{q_1^2+q_2^2} \ll 1.$$

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